1B45 Mathematical Methods Problem Class 3 2005/2006

Week starting Monday 14th. November

Solutions

1.

If the radius of the circle is r then its circumference is $2\pi r$ and the length of wire left over for the square is $40-2\pi r$ so that each side of the square is of length $10-\frac{1}{2}\pi r$. Hence the combined area is given by

$$A = \pi r^2 + (10 - \frac{1}{2}\pi r)^2 \quad \text{and} \quad \frac{dA}{dr} = 2\pi r + 2(10 - \frac{1}{2}\pi r)(-\frac{1}{2}\pi) = 2\pi r\left(1 + \frac{\pi}{4}\right) - 10\pi \ .$$

$$\text{Setting} \quad \frac{dA}{dr} = 0 \quad \text{we find} \quad r\left(1 + \frac{\pi}{4}\right) = 5 \quad , \quad r = 2.8 \quad \text{and} \quad A = 56cm^2 \ .$$

$$\text{Further we find that} \quad \frac{d^2A}{dr^2} = 2\pi\left(1 + \frac{\pi}{4}\right) > 0 \ .$$

so it is the minimum area that we have found!!

This is somewhat of an ill posed problem - the length of the wire imposes a discontinuity. If the maximum were to be found between $r = 0$ and $r = 40/2\pi$ then the calculus would have found it. We must conclude therefore that the maximum must be at one end or other of the range of r. In fact A takes the largest value when all the wire is used to make the circle. Then r is 6.37 cm and A is 127 cm^2.

2.

Note the following partial derivatives can be obtained directly from the transformation equations. The approach adopted here always works for less straight forward transformations.

Since $u_1^2 - u_2^2 = x$ and $2u_1u_2 = y$ then taking differentials we obtain

$$2u_1du_1 - 2u_2du_2 = dx$$
$$2u_2du_1 + 2u_1du_2 = dy \ .$$

For partial derivatives w. r. t. u_1 we set $du_2 = 0$ in the above and vice versa for derivatives w. r. t. u_2.

$$\text{We find} \quad \frac{\partial x}{\partial u_1} = 2u_1 \quad , \quad \frac{\partial y}{\partial u_1} = 2u_2 \quad , \quad \frac{\partial x}{\partial u_2} = -2u_2 \quad \text{and} \quad \frac{\partial y}{\partial u_2} = 2u_1 \ .$$

Squaring both of the coordinate transformation equations and adding yields

$$x^2 + y^2 = (u_1^2 + u_2^2)^2 \quad \text{and from the above} \quad \sqrt{\left(\frac{\partial x}{\partial u_1}\right)^2 + \left(\frac{\partial y}{\partial u_1}\right)^2} = 2\sqrt{(u_1^2 + u_2^2)} \ .$$

$$\text{Hence} \quad \sqrt{\left(\frac{\partial x}{\partial u_1}\right)^2 + \left(\frac{\partial y}{\partial u_1}\right)^2} = 2\sqrt{(u_1^2 + u_2^2)} = 2(x^2 + y^2)^{\frac{1}{4}}$$

3.

We find

$$\frac{\partial f}{\partial x} = (3x^2 - 2x^4)e^{(-x^2-y^2)} = 0 \text{ and } \frac{\partial f}{\partial y} = -2yx^3 e^{(-x^2-y^2)} = 0 \ .$$

For the second equation we find that $f_y = 0$ at $x = 0$ and $y = 0$. From the first equation we find that $f_x = 0$ when $x = 0$ or $x = \pm\sqrt{3/2}$. Hence the stationary points are at $(0, 0)$, $(\sqrt{3/2}, 0))$ and $(-\sqrt{3/2}, 0)$. At these points **both** f_x and f_y vanish. The second derivatives are given by

$$f_{xx} = (4x^5 - 14x^3 + 6x)e^{(-x^2-y^2)} \ , \ f_{yy} = x^3(4y^2 - 2)e^{(-x^2-y^2)} \text{ and } f_{xy} = 2x^2 y(2x^2 - 3)e^{(-x^2-y^2)} \ .$$

For the coordinate $(0, 0)$, $f_{xx} = 0$, $f_{yy} = 0$ and $f_{xy} = 0$ and the nature of this point is indeterminate!!

For the coordinates $(\pm\sqrt{3/2}, 0)$, $f_{xx} = \mp 6\sqrt{3/2}e^{-3/2}$, $f_{yy} = \mp 3\sqrt{3/2}e^{-3/2}$ and $f_{xy} = 0$.

Applying the criteria we find that the stationary point at $(\sqrt{3/2}, 0)$ is a maximum and that at $(-\sqrt{3/2}, 0)$ a minimum.

4.

The problem here is to maximise the volume of the rectangular parallelipiped $f = 8xyz$ subject to the ellipsoidal constraint equation ϕ

We have

$$F(x, y, z) = f + \lambda\phi = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) \ .$$

The three partial derivatives of F are now set to zero; one so that the Lagrange multiplier eliminates the dependent differential, the other two then expressing the stationary conditions from the independent differentials.

$$\text{We find } 8yz + \lambda\frac{2x}{a^2} = 0 \ , \ 8xz + \lambda\frac{2y}{b^2} = 0 \text{ and } 8xy + \lambda\frac{2z}{c^2} = 0 \ .$$

From these we readily find

$$3 \times 8xyz + 2\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 0 \text{ or } 24xyz + 2\lambda = 0 \text{ and } \lambda = -12xy \ .$$

Putting this back into the partial derivative equations which were set to zero we find

$$x^2 = \frac{1}{3}a^2 \ , \ y^2 = \frac{1}{3}b^2 \ , \ z^2 = \frac{1}{3}c^2 \text{ and } 8xyz = \frac{8abc}{3\sqrt{3}} \ .$$