1 MT1 Homework Problem Solutions: 7

1)

$$I_n = \int x^n e^{-x} \mathrm{dx}$$

Integrate this by parts, taking $u = x^n$, so $u' = nx^{n-1}$, $v' = e^{-x}$, so $v = -e^{-x}$. Hence

$$\begin{aligned} I_n &= -x^n e^{-x} + n \int x^{n-1} e^{-x} \mathrm{dx} \\ &= -x^n e^{-x} + nI_{n-1} \end{aligned}$$

So,

$$\begin{aligned} I_3 &= -x^3 e^{-x} + 3\left[-x^2 e^{-x} + 2\left(-xe^{-x} - e^{-x}\right)\right] + C \\ &= -e^{-x}\left[x^3 + 3x^2 + 6x + 6\right] + C \end{aligned}$$

[4 mark(s)]

2)

$$\begin{aligned} \langle y \rangle &= \frac{1}{\frac{1}{2} - 0} \int_0^{1/2} 1 + \cos(\pi x) \mathrm{dx} \\ &= 2\left[x + \frac{\sin(\pi x)}{\pi}\right]_0^{1/2} \\ &= 2\left[\left(\frac{1}{2} + \frac{\sin(\pi/2)}{\pi}\right) - \left(0 + \frac{\sin(0)}{\pi}\right)\right] \\ &= 1 + \frac{2}{\pi} \end{aligned}$$

[3 mark(s)]

3)

$$
\begin{aligned}
A &= \int\limits_{x=0,\theta=\pi/4}^{x=2,\theta=0} y\,\mathrm{dx}, \\
\frac{\mathrm{dx}}{\mathrm{d}\theta} &= -2\sin\theta, \\
\mathrm{dx} &= -2\sin\theta \mathrm{d}\theta \\
A &= -2\int\limits_{\theta=\pi/4}^{0} \sin^2\theta\,\mathrm{d}\theta, \\
&= \int\limits_{\theta=\pi/4}^{0} \cos(2\theta) - 1\,\mathrm{d}\theta, \\
&= \left[\frac{\sin(2\theta)}{2} - \theta\right]_{\theta=\pi/4}^{0}, \\
&= \pi/4 - \frac{1}{2}\sin(\pi/2), \\
&= 0.2854\,(4d.p.).
\end{aligned}
$$

[5 mark(s)]

4)

$$
\begin{aligned}
V &= V_0\cos(\omega t + \phi) \\
RMS &= \sqrt{\langle V^2 \rangle} \\
&= \sqrt{\frac{\omega}{2\pi}V_0^2 \int\limits_{-\phi/\omega}^{(2\pi-\phi)/\omega} \cos^2(\omega t + \phi)\,\mathrm{dt}}
\end{aligned}
$$

Recall that $\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$, so the integral is

$$
\begin{aligned}
I &= \int\limits_{-\phi/\omega}^{(2\pi-\phi)/\omega} \cos^2(\omega t+\phi)\,\mathrm{dt} \\
&= \frac{1}{2}\int\limits_{-\phi/\omega}^{(2\pi-\phi)/\omega} 1+\cos(2\omega t+2\phi)\,\mathrm{dt} \\
&= \frac{1}{2}\left[t+\frac{\sin(2\omega t+2\phi)}{2\omega}\right]_{-\phi/\omega}^{(2\pi-\phi)/\omega} \\
&= \frac{1}{2}\left[\left(\frac{(2\pi-\phi)}{\omega}+\frac{\sin(4\pi)}{2\omega}\right)-\left(\frac{\sin(0)}{2\omega}-\frac{\phi}{\omega}\right)\right] \\
&= \frac{1}{2}\left[\frac{(2\pi-\phi)}{\omega}+\frac{\phi}{\omega}\right] \\
&= \frac{\pi}{\omega}
\end{aligned}
$$

so

$$
\begin{aligned}
RMS &= \sqrt{\frac{\omega}{2\pi}V_0^2\times\left(\frac{\pi}{\omega}\right)} \\
&= \frac{V_0}{\sqrt{2}}
\end{aligned}
$$

[8 mark(s)]

5)

i)

$$
\begin{aligned}
I &= \int\limits_0^\infty e^{-2x}\mathrm{dx} \\
&= \left[-\frac{1}{2}e^{-2x}\right]_0^\infty \\
&= \left[0+\frac{1}{2}\right] \\
&= \frac{1}{2}
\end{aligned}
$$

[2 mark(s)]

ii)

$$I = \int_1^e \ln|x| \mathrm{dx} \tag{1}$$

Let $z = \ln|x|$, so $z' = \frac{1}{x}$ and $\mathrm{dx} = x\mathrm{dz} = \mathrm{e^z dz}$.

$$I' = \int_{x=1}^{x=e} ze^z \mathrm{dz} \tag{2}$$

which can be integrated by parts to give $\int \ln|x|\mathrm{dx} = x\ln|x| - x + C$, so the corresponding definite integral becomes

$$\begin{aligned} I &= [x\ln|x| - x]_1^e \\ &= (e\ln e - e) - (\ln 1 - 1) \\ &= 1 \end{aligned}$$

[3 mark(s)]

6)

$$\langle y\rangle = \frac{1}{1-0}\int_0^1 xe^{-2x}\,\mathrm{dx}$$

which can be integrated by parts taking $u = x$ and $v' = e^{-2x}$; so $u' = 1$ and

$v = -e^{-2x}/2$. Thus

$$\begin{aligned}
\langle y \rangle &= \left[-\frac{xe^{-2x}}{2} \right]_0^1 + \frac{1}{2}\int_0^1 e^{-2x}\,\mathrm{dx} \\
&= \left[-\frac{xe^{-2x}}{2} - \frac{1}{4}e^{-2x} \right]_0^1 \\
&= -e^{-2}\left(\frac{1}{2} + \frac{1}{4} \right) + \left(\frac{1}{4} \right) \\
&= \frac{1}{4} - \frac{3e^{-2}}{4} \\
&= 0.1485 (4 d.p.).
\end{aligned}$$

[4 mark(s)]