1 MT1 Homework Problem Solutions: 5

1)

i)

$$\lim_{x\to\infty}\left(\frac{3x^3+4x+1}{7x^3+2x^2+1}\right)=\lim_{x\to\infty}\left(\frac{3+4/x^2+x^3}{7+2/x+1/x^3}\right)=\frac{3}{7}$$

[1 mark(s)]

ii)

$$\lim_{x\to\infty}\left(\frac{x^5+4x^3+x}{3x^5+2x}\right)=\lim_{x\to\infty}\left(\frac{1+4/x^2+1/x^4}{3+2/x^4}\right)=\frac{1}{3}$$

[1 mark(s)]

2)

$$(1+x)^n = \sqrt{0.8},$$

so $x=-0.2,$ and $n=1/2$

$$\begin{aligned}(1+x)^n &\simeq 1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)x^3}{3!},\\ &= 1+\frac{1}{2}(-0.2)+\frac{1}{2}\times\frac{-1}{2}\frac{(-0.2)^2}{2!}+\frac{1}{2}\times\frac{-1}{2}\times\frac{-3}{2}\frac{(-0.2)^3}{3!}\\ &= 1-0.1-0.005-0.0005\\ &= 0.8945(4d.p.)\end{aligned}$$

Compare this result to a calculator (which gives $\sqrt{0.8}=0.8944(4d.p.)$) and the difference is at a level of 7.3×10^{-5}.

[4 mark(s)]

3) Determine a Taylor series expansion for $\cos(2x)$ about the point $x=\pi/8$, hence determine a series expression for $\cos^2(2x)$.

$$\begin{aligned}f(x) &= \cos(2x)\\ &= f_a+f'_a(x-a)+\frac{f''_a(x-a)^2}{2!}+\frac{f'''_a(x-a)^3}{3!}+\ldots\end{aligned}$$

where $a = \pi/8$, and

$$\begin{aligned}
f(x) &= \cos(2x); \text{so } f(\pi/8) = \frac{1}{\sqrt{2}} \\
f'(x) &= -2\sin(2x); \text{so } f'(\pi/8) = -\frac{2}{\sqrt{2}} \\
f''(x) &= -4\cos(2x); \text{so } f''(\pi/8) = -\frac{4}{\sqrt{2}} \\
f'''(x) &= 8\sin(2x); \text{so } f'''(\pi/8) = \frac{8}{\sqrt{2}}
\end{aligned}$$

Using these results we obtain

$$\cos(2x) = \frac{1}{\sqrt{2}}\left[1 - 2\left(x - \frac{\pi}{8}\right) - 2\left(x - \frac{\pi}{8}\right)^2 + \frac{4\left(x - \frac{\pi}{8}\right)^3}{3} + \ldots\right]$$

So we can write

$$\begin{aligned}
\cos^2(2x) &= \left(\frac{1}{\sqrt{2}}\right)^2 \left[1 - 2\left(x - \frac{\pi}{8}\right) - 2\left(x - \frac{\pi}{8}\right)^2 + \frac{4\left(x - \frac{\pi}{8}\right)^3}{3} + \ldots\right]^2 \\
&= \frac{1}{2}\left[1 - 4\left(x - \frac{\pi}{8}\right) + 8\left(x - \frac{\pi}{8}\right)^3 - \frac{4}{3}\left(x - \frac{\pi}{8}\right)^4 + \ldots\right] \\
&= \left[\frac{1}{2} - 2\left(x - \frac{\pi}{8}\right) + 4\left(x - \frac{\pi}{8}\right)^3 - \frac{2}{3}\left(x - \frac{\pi}{8}\right)^4 + \ldots\right]
\end{aligned}$$

[6 mark(s)]

4)

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \\
e^{Ax} &= 1 + Ax + \frac{A^2x^2}{2!} + \frac{A^3x^3}{3!} + \frac{A^4x^4}{4!} + \ldots \\
e^{2x} &= 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{4!} + \ldots \\
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \\
\sin(\pi x) &= \pi x - \frac{\pi^3x^3}{3!} + \frac{\pi^5x^5}{5!} - \frac{\pi^7x^7}{7!} + \ldots
\end{aligned}$$

So the Maclaurin series expansions for $e^{2x}\sin(\pi x)$ is given by

$$\begin{aligned} e^{2x}\sin(\pi x) &= \left(1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \ldots\right) \times \left(\pi x - \frac{\pi^3 x^3}{3!} + \frac{\pi^5 x^5}{5!} - \frac{\pi^7 x^7}{7!} + \ldots\right). \\ &= \pi x + 2\pi x^2 + x^3\left(2\pi - \frac{\pi^3}{6}\right) + x^4\left(-\frac{\pi^3}{3}\right) + \ldots \end{aligned}$$

[4 mark(s)]

5) Write down the Maclaurin series expansion for $f(x) = e^{-\gamma x}$. Using the first four terms of the expansion estimate $f(x)$ when $x = \ln 2/\gamma$.

$$\begin{aligned} e^{x} &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \\ e^{-\gamma x} &= 1 - \gamma x + \frac{\gamma^2 x^2}{2!} - \frac{\gamma^3 x^3}{3!} + \frac{\gamma^4 x^4}{4!} + \ldots \\ e^{-\gamma \ln 2/\gamma} &= 1 - \ln 2 + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \frac{(\ln 2)^4}{4!} + \ldots \\ &= 1 - 0.693147 + 0.240227 - 0.055504 \\ &= 0.49158\,(5d.p.). \end{aligned}$$

[4 mark(s)]