1 MT1 Homework Problem Solutions: 4

1)

i) $z = e^x + y^2$, so

$$\begin{aligned}
\frac{\partial \mathrm{z}}{\partial \mathrm{x}} &= e^x, \\
\frac{\partial^2 \mathrm{z}}{\partial \mathrm{x}^2} &= e^x, \\
\frac{\partial \mathrm{z}}{\partial \mathrm{y}} &= 2y, \\
\frac{\partial^2 \mathrm{z}}{\partial \mathrm{y}^2} &= 2, \\
\frac{\partial^2 \mathrm{z}}{\partial \mathrm{x} \partial \mathrm{y}} &= \frac{\partial^2 \mathrm{z}}{\partial \mathrm{y} \partial \mathrm{x}} = 0.
\end{aligned}$$

[5 mark(s)]

ii) $z = y \sin(x) + y^2$.

$$\begin{aligned}
\frac{\partial \mathrm{z}}{\partial \mathrm{x}} &= y \cos(x), \\
\frac{\partial^2 \mathrm{z}}{\partial \mathrm{x}^2} &= -y \sin(x), \\
\frac{\partial \mathrm{z}}{\partial \mathrm{y}} &= \sin(x) + 2y, \\
\frac{\partial^2 \mathrm{z}}{\partial \mathrm{y}^2} &= 2, \\
\frac{\partial^2 \mathrm{z}}{\partial \mathrm{x} \partial \mathrm{y}} &= \frac{\partial^2 \mathrm{z}}{\partial \mathrm{y} \partial \mathrm{x}} = \cos(x).
\end{aligned}$$

[5 mark(s)]

iii) $z = e^{-ET}$.

$$\begin{aligned}
\frac{\partial \mathrm{z}}{\partial \mathrm{T}} &= -Ee^{-ET}, \\
\frac{\partial^2 \mathrm{z}}{\partial \mathrm{T}^2} &= E^2e^{-ET}, \\
\frac{\partial \mathrm{z}}{\partial \mathrm{E}} &= -Te^{-ET}, \\
\frac{\partial^2 \mathrm{z}}{\partial \mathrm{E}^2} &= T^2e^{-ET}, \\
\frac{\partial^2 \mathrm{z}}{\partial \mathrm{E}\partial \mathrm{T}} &= \frac{\partial^2 \mathrm{z}}{\partial \mathrm{T}\partial \mathrm{E}} = (ET-1)e^{-ET}.
\end{aligned}$$

[5 mark(s)]

iv) $\sin(\alpha x + \beta t)$, where α and β are constants.

$$\begin{aligned}
\frac{\partial \mathrm{z}}{\partial \mathrm{x}} &= \alpha\cos(\alpha x + \beta t), \\
\frac{\partial^2 \mathrm{z}}{\partial \mathrm{x}^2} &= -\alpha^2\sin(\alpha x + \beta t), \\
\frac{\partial \mathrm{z}}{\partial \mathrm{t}} &= \beta\cos(\alpha x + \beta t), \\
\frac{\partial^2 \mathrm{z}}{\partial \mathrm{t}^2} &= -\beta^2\sin(\alpha x + \beta t), \\
\frac{\partial^2 \mathrm{z}}{\partial \mathrm{x}\partial \mathrm{t}} &= \frac{\partial^2 \mathrm{z}}{\partial \mathrm{t}\partial \mathrm{x}} = -\alpha\beta\sin(\alpha x + \beta t).
\end{aligned}$$

[5 mark(s)]

2) Show that $z = e^x(x\cos y + y\sin x)$ satisfies

$$\frac{\partial^2 \mathrm{z}}{\partial \mathrm{x}^2} + \frac{\partial^2 \mathrm{z}}{\partial \mathrm{y}^2} = 2e^x(\cos y + y\cos x).$$

$$
\begin{aligned}
\frac{\partial \mathrm{z}}{\partial \mathrm{x}} &= \cos(y)e^x(1+x) + y(-\cos(x)e^x + \sin(x)e^x), \\
\frac{\partial^2 \mathrm{z}}{\partial \mathrm{x}^2} &= (2+x)\cos(y)e^x + 2ye^x\cos(x), \\
\frac{\partial \mathrm{z}}{\partial \mathrm{y}} &= -xe^x\sin(y) + \sin(x), \\
\frac{\partial^2 \mathrm{z}}{\partial \mathrm{y}^2} &= -xe^x\cos(y),
\end{aligned}
$$

so,

$$
\begin{aligned}
\frac{\partial^2 \mathrm{z}}{\partial \mathrm{x}^2} + \frac{\partial^2 \mathrm{z}}{\partial \mathrm{y}^2} &= 2\cos(y)e^x + 2ye^x\cos(x) \\
&= 2e^x(\cos y + y\cos x).
\end{aligned}
$$

[5 mark(s)]

3) Two masses m_1 and m_2 are placed near each other in a Cavendish experiment, whose aim is to measure the gravitational constant G. The position of the first mass is given by x_1, and the position of the second mass is given by x_2 on some scale. Calculate the fractional change in the force between the masses $\delta F/F$ due small changes in x_1 and x_2 given that

$$
\begin{aligned}
F &= \frac{Gm_1m_2}{(x_1-x_2)^2}, \\
\frac{\partial \mathrm{F}}{\partial \mathrm{x}_1} &= -\frac{2Gm_1m_2}{(x_1-x_2)^3}, \\
\frac{\partial \mathrm{F}}{\partial \mathrm{x}_2} &= \frac{2Gm_1m_2}{(x_1-x_2)^3}, \\
\delta F &= \frac{2Gm_1m_2}{(x_1-x_2)^3}\left(\delta x_2 - \delta x_1\right), \\
\frac{\delta F}{F} &= \frac{2(\delta x_2 - \delta x_1)}{(x_1-x_2)}.
\end{aligned}
$$

[4 mark(s)]

4) Given that $z = xy^2 + x$, where $x = R\cos\theta$ and $y = R\sin\theta$, calculate $\frac{\mathrm{dz}}{\mathrm{d}\theta}$ in terms of θ.

$$
\begin{aligned}
\frac{\mathrm{dz}}{\mathrm{d}\theta} &= \frac{\partial \mathrm{z}}{\partial \mathrm{x}}\frac{\mathrm{dx}}{\mathrm{d}\theta} + \frac{\partial \mathrm{z}}{\partial \mathrm{y}}\frac{\mathrm{dy}}{\mathrm{d}\theta}, \\
&= (y^2+1)(-R\sin\theta) + 2xyR\cos\theta, \\
&= (R^2\sin^2\theta + 1)(-R\sin\theta) + 2R^3\cos^2\theta\sin\theta. \\
&= 2R^3\cos^2\theta\sin\theta - R^3\sin^3\theta - R\sin\theta.
\end{aligned}
$$

[4 mark(s)]

5) Find $\frac{\partial \mathrm{z}}{\partial\theta}$ and $\frac{\partial \mathrm{z}}{\partial\phi}$ in terms of x and t for $z = \sin(kx - \omega t)$, where $\theta = x + t^2$ and $\phi = x^2 - t$. k and ω are constants.

$$
\begin{aligned}
\frac{\partial \mathrm{z}}{\partial\theta} &= \frac{\partial \mathrm{z}}{\partial \mathrm{x}}\frac{\partial \mathrm{x}}{\partial\theta} + \frac{\partial \mathrm{z}}{\partial \mathrm{t}}\frac{\partial \mathrm{t}}{\partial\theta}, \\
\frac{\partial \mathrm{z}}{\partial\phi} &= \frac{\partial \mathrm{z}}{\partial \mathrm{x}}\frac{\partial \mathrm{x}}{\partial\phi} + \frac{\partial \mathrm{z}}{\partial \mathrm{t}}\frac{\partial \mathrm{t}}{\partial\phi},
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{\partial \mathrm{z}}{\partial \mathrm{x}} &= k\cos(kx - \omega t), \\
\frac{\partial \mathrm{z}}{\partial \mathrm{t}} &= -\omega\cos(kx - \omega t), \\
\frac{\partial \mathrm{x}}{\partial\theta} &= 1, \\
\frac{\partial \mathrm{t}}{\partial\theta} &= \frac{1}{2}\left(\theta - x\right)^{-1/2}, \mathrm{as}\, t = \sqrt{\theta - x} \\
\frac{\partial \mathrm{x}}{\partial\phi} &= \frac{1}{2}\left(\phi + t\right)^{-1/2}, \mathrm{as}\, x = \sqrt{\phi + t} \\
\frac{\partial \mathrm{t}}{\partial\phi} &= -1.
\end{aligned}
$$

Note here that in general

$$
\begin{aligned}
&\frac{\partial \mathrm{t}}{\partial\theta} \neq 1/\frac{\partial\theta}{\partial \mathrm{t}}, \\
&\frac{\partial \mathrm{x}}{\partial\phi} \neq 1/\frac{\partial\phi}{\partial \mathrm{x}},
\end{aligned}
$$

and so on. If you missed this point, then your answer will be incorrect.
So

$$\begin{aligned}
\frac{\partial \mathrm{z}}{\partial \theta} &= k\cos(kx-\omega t) - \frac{\omega\cos(kx-\omega t)}{2\sqrt{\theta - x}},\\
\frac{\partial \mathrm{z}}{\partial \phi} &= \frac{k\cos(kx-\omega t)}{2\sqrt{\phi + t}} + \omega\cos(kx-\omega t).
\end{aligned}$$

The RHS of this final result can be trivially re-written in terms of x and t.

[8 mark(s)]

6) Given that $z = \ln|x|\sin(y)$ where $x = uv^2$ and $y = v\sin u$, calculate $\frac{\partial \mathrm{z}}{\partial \mathrm{u}}$ and $\frac{\partial \mathrm{z}}{\partial \mathrm{v}}$.

$$\begin{aligned}
\frac{\partial \mathrm{z}}{\partial \mathrm{u}} &= \frac{\partial \mathrm{z}}{\partial \mathrm{x}}\frac{\partial \mathrm{x}}{\partial \mathrm{u}} + \frac{\partial \mathrm{z}}{\partial \mathrm{y}}\frac{\partial \mathrm{y}}{\partial \mathrm{u}},\\
\frac{\partial \mathrm{z}}{\partial \mathrm{v}} &= \frac{\partial \mathrm{z}}{\partial \mathrm{x}}\frac{\partial \mathrm{x}}{\partial \mathrm{v}} + \frac{\partial \mathrm{z}}{\partial \mathrm{y}}\frac{\partial \mathrm{y}}{\partial \mathrm{v}},
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial \mathrm{z}}{\partial \mathrm{x}} &= \frac{\sin y}{x},\\
\frac{\partial \mathrm{z}}{\partial \mathrm{y}} &= \ln|x|\cos y,\\
\frac{\partial \mathrm{x}}{\partial \mathrm{u}} &= v^2\\
\frac{\partial \mathrm{x}}{\partial \mathrm{v}} &= 2uv,\\
\frac{\partial \mathrm{y}}{\partial \mathrm{u}} &= v\cos u,\\
\frac{\partial \mathrm{y}}{\partial \mathrm{v}} &= \sin u.
\end{aligned}$$

So

$$\begin{aligned}
\frac{\partial \mathrm{z}}{\partial \mathrm{u}} &= \frac{v^2\sin y}{x} + \ln|x|v\cos u\cos y,\\
&= \frac{\sin(v\sin u)}{u} + v\ln|uv^2|\cos u\cos(v\sin u),\\
\frac{\partial \mathrm{z}}{\partial \mathrm{v}} &= \frac{2uv\sin y}{x} + \ln|x|\cos y\sin u,\\
&= \frac{2\sin(v\sin u)}{v} + \ln|uv^2|\cos(v\sin u)\sin u.
\end{aligned}$$

[8 mark(s)]