1 MT1 Homework Problem Solutions: 2

1)

i) $\int \alpha\cos(\pi x)\text{dx} = \frac{\alpha}{\pi}\sin(\pi x) + C$

[1 mark(s)]

ii) $\int \frac{1}{\pi}e^{\pi x}\text{dx} = \frac{1}{\pi^2}e^{\pi x} + C$

[1 mark(s)]

iii) $\int x\cos(\pi x)\text{dx}$; let $u = x$, so $v = \frac{1}{\pi}\sin(\pi x)$

$$\begin{aligned}\int x\cos(\pi x)\text{dx} &= \frac{x}{\pi}\sin(\pi x) - \frac{1}{\pi}\int \sin(\pi x)\text{dx} \\ &= \frac{x}{\pi}\sin(\pi x) + \frac{1}{\pi^2}\cos(\pi x) + C\end{aligned}$$

[3 mark(s)]

iv)

$$\begin{aligned}\int \tan(\pi x)\text{dx} &= \int \frac{\sin(\pi x)}{\cos(\pi x)}\text{dx} \\ \text{let } u &= \cos(\pi x) \\ \text{so } \frac{\text{du}}{\text{dx}} &= -\pi\sin(\pi x) \\ \int \frac{\sin(\pi x)}{\cos(\pi x)}\text{dx} &= -\frac{1}{\pi}\int \frac{\text{du}}{u} \\ &= -\frac{1}{\pi}\ln|\cos(\pi x)| + C\end{aligned}$$

[3 mark(s)]

2) Integrate $e^{\pi x}\cos(\pi x)$ with respect to x, by parts. Let $u = e^{\pi x}$, $v' = \cos(\pi x)$, so $u' = \pi e^{\pi x}$ and $v = \sin(\pi x)/\pi$

$$\int e^{\pi x}\cos(\pi x)\text{dx} = \frac{e^{\pi x}\sin(\pi x)}{\pi} - \int e^{\pi x}\sin(\pi x)\text{dx}$$

where we can integrate $e^{\pi x}\sin(\pi x)$ taking $u = e^{\pi x}$, $v' = \sin(\pi x)$, so $u' = \pi e^{\pi x}$

and $v = -\cos(\pi x)/\pi$

$$\begin{aligned}\int e^{\pi x}\cos(\pi x)\mathrm{dx} &= \frac{e^{\pi x}\sin(\pi x)}{\pi} - \left[-\frac{e^{\pi x}\cos(\pi x)}{\pi} + I\right] + C \\ \int e^{\pi x}\cos(\pi x)\mathrm{dx} &= \frac{e^{\pi x}[\sin(\pi x) + \cos(\pi x)]}{2\pi} + C\end{aligned}$$

[4 mark(s)]

3) Integrate the following.

i) $x\ln|x|$. Let $u = x$, $v' = \ln x$, so $u' = 1$ and $v = x\ln|x| - x$.

$$\begin{aligned}\int x\ln|x|\mathrm{dx} &= x^2\ln|x| - x^2 - I + \int x\mathrm{dx} \\ &= \frac{x^2\ln|x|}{2} - \frac{x^2}{4} + C\end{aligned}$$

[2 mark(s)]

ii) $\sin^2(x - \pi/6) = \frac{1}{2}[1 - \cos(2x - \pi/3)]$, so

$$\frac{1}{2}\int[1 - \cos(2x - \pi/3)]\,\mathrm{dx} = \frac{1}{2}\left[x - \frac{1}{2}\sin(2x - \pi/3)\right] + C$$

[2 mark(s)]

iii) $x\sin(\pi x)$; let $u = x$, so $u' = 1$, $v' = \sin(\pi x)$, and $v = -\cos(\pi x)/\pi$, so

$$\begin{aligned}\int x\sin(\pi x)\mathrm{dx} &= -\frac{x\cos(\pi x)}{\pi} + \frac{1}{\pi}\int\cos(\pi x)\mathrm{dx} \\ &= -\frac{x\cos(\pi x)}{\pi} + \frac{\sin(\pi x)}{\pi^2} + C\end{aligned}$$

[2 mark(s)]

4) Evaluate $\int\frac{1}{(x-1)(2x+1)}\mathrm{dx}$. We can separate the quotient into partial fractions as follows:

$$\begin{aligned}\frac{1}{(x-1)(2x+1)} &= \frac{A}{x-1} + \frac{B}{2x+1} \\ &= \frac{A(2x+1) + B(x-1)}{(x-1)(2x+1)}\end{aligned}$$

so $A = 1/3$ and $B = -2/3$. So

$$\begin{aligned}
\int \frac{1}{(x-1)(2x+1)}\mathrm{dx} &= \int \frac{A}{x-1}\mathrm{dx} + \int \frac{B}{2x+1}\mathrm{dx} \\
&= \frac{1}{3}\int \frac{1}{x-1}\mathrm{dx} - \frac{1}{3}\int \frac{2}{2x+1}\mathrm{dx} \\
&= \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|2x+1| + C
\end{aligned}$$

[4 mark(s)]

5) Evaluate

i) $\int\limits_0^{\pi} \cos^2 x\mathrm{dx}$; recall that $\cos^2 x = \frac{1}{2}(1+\cos 2x)$, so

$$\begin{aligned}
\int\limits_0^{\pi} \cos^2 x\mathrm{dx} &= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi} \\
&= \frac{\pi}{2}
\end{aligned}$$

[2 mark(s)]

ii) $\int\limits_0^{1/2} x\sin\pi x\mathrm{dx}$

$$\begin{aligned}
\int\limits_0^{1/2} x\sin(\pi x)\mathrm{dx} &= \left[-\frac{x\cos(\pi x)}{\pi} + \frac{\sin(\pi x)}{\pi^2}\right]_0^{1/2} \\
&= \frac{\sin(\pi/2)}{\pi^2} \\
&= \frac{1}{\pi^2}
\end{aligned}$$

[2 mark(s)]

6)

$$\begin{aligned}\int_0^{+\infty} e^{-x}\mathrm{dx} &= \left[-e^{-x}\right]_0^{+\infty} \\ &= [0+1] \\ &= 1\end{aligned}$$

[2 mark(s)]