1 MT1 Homework Problem Solutions: 1

1)

i) $\frac{\mathrm{d}}{\mathrm{dx}}\left[\sin(x)\cos(\pi x)\right] = \cos(x)\cos(\pi x) - \pi\sin(x)\sin(\pi x)$

[1 mark(s)]

ii) $\frac{\mathrm{d}}{\mathrm{dx}}\left[(ax-b)^n\right] = an(ax-b)^{n-1}$

[1 mark(s)]

iii) $\frac{\mathrm{d}}{\mathrm{dx}}\left[\sin^2(\pi x)\right] = 2\pi\cos(\pi x)\sin(\pi x)$

[1 mark(s)]

iv) $\frac{\mathrm{d}}{\mathrm{dx}}\left[\ln[\cos(x)]\right]$, let $u = \cos(x)$, so $\frac{\mathrm{du}}{\mathrm{dx}} = -\sin(x)$, and $\frac{\mathrm{d}}{\mathrm{du}}\ln \mathrm{u} = \frac{1}{\mathrm{u}}$. So

$$\begin{aligned}\frac{\mathrm{dy}}{\mathrm{dx}} &= \frac{\mathrm{du}}{\mathrm{dx}}\frac{\mathrm{d}}{\mathrm{du}}\ln \mathrm{u}, \\ &= \frac{-\sin(x)}{\cos(x)}, \\ &= -\tan(x).\end{aligned}$$

[2 mark(s)]

v) $\frac{\mathrm{d}}{\mathrm{dx}}\left[\log_{10}(x^2+1)\right] = \frac{2x}{(x^2+1)\ln 10}$

[1 mark(s)]

vi) $\frac{\mathrm{d}}{\mathrm{dx}}\left[e^{ax}\sin(x)\right] = e^{ax}(a\sin(x) + \cos(x))$

[1 mark(s)]

vii) $\frac{\mathrm{d}}{\mathrm{dx}}\left[\frac{x^2}{(1+x)^2}\right] = \frac{2x(1+x)^2 - 2x^2(1+x)}{(1+x)^4} = \frac{2x}{(1+x)^3}$,

[1 mark(s)]

viii) $\frac{\mathrm{d}}{\mathrm{dx}}\left[\sin^3\left(\frac{x}{2\pi}\right)\right]$,

$$\begin{aligned}\text{let } u &= \sin(x/2\pi), \\ \text{so } \frac{\mathrm{du}}{\mathrm{dx}} &= \frac{1}{2\pi}\cos(x/2\pi), \\ \frac{\mathrm{d}}{\mathrm{du}}(u^3) &= 3u^2.\end{aligned}$$

we can now substitute back for x, in order to obtain the following

$$\frac{\mathrm{d}}{\mathrm{dx}}\left[\sin^3\left(\frac{x}{2\pi}\right)\right] = \frac{3}{(2\pi)}\cos\left(\frac{x}{2\pi}\right)\sin^2\left(\frac{x}{2\pi}\right)$$

[3 mark(s)]

ix) $\frac{\mathrm{d}}{\mathrm{dx}}\left[\frac{e^x\sin(x)}{x^2\cos(x)}\right]$, here we have $u = e^x\sin(x)$, and $v = x^2\cos(x)$, and we must differentiate by parts, so

$$\begin{aligned}\frac{\mathrm{du}}{\mathrm{dx}} &= \frac{\mathrm{d}}{\mathrm{dx}}[e^x\sin(x)] = e^x[\sin(x)+\cos(x)]\\ \frac{\mathrm{dv}}{\mathrm{dx}} &= \frac{\mathrm{d}}{\mathrm{dx}}[x^2\cos(x)] = 2x\cos(x) - x^2\sin(x)\end{aligned}$$

so

$$\begin{aligned}\frac{\mathrm{dy}}{\mathrm{dx}} &= \frac{x^2\cos(x)e^x[\sin(x)+\cos(x)] - e^x\sin(x)[2x\cos(x)-x^2\sin(x)]}{x^4\cos^2(x)}\\ &= \frac{(x^2-2x)e^x\cos(x)\sin(x) + x^2\cos^2(x)e^x + x^2e^x\sin^2(x)}{x^4\cos^2(x)}\end{aligned}$$

[4 mark(s)]

x) $\frac{\mathrm{d}}{\mathrm{dx}}[\ln(\pi x)] = \frac{1}{x}$. We can check this by using the chain rule, were we let $u = \pi x$, so that we calculate $\frac{\mathrm{d}}{\mathrm{dx}}[\ln(\pi x)] = \frac{\mathrm{d}}{\mathrm{du}}[\ln u]\frac{\mathrm{du}}{\mathrm{dx}}$. Here $\frac{\mathrm{du}}{\mathrm{dx}} = \pi$, and $\frac{\mathrm{d}}{\mathrm{du}}[\ln u] = \frac{1}{u}$. If we substitute back to x, then we obtain $\frac{\mathrm{d}}{\mathrm{dx}}[\ln(\pi x)] = \frac{1}{x}$ as required.

[1 mark(s)]

2) Evaluate by implicit differentiation

i) $\frac{\mathrm{dy}}{\mathrm{dx}}$ when $x^3 + y^2 = 1$, so we can differentiate both sides of this equation as follows

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dx}}\left[x^3+y^2\right] &= \frac{\mathrm{d}}{\mathrm{dx}}[1]\\ 3x^2 + 2y\frac{\mathrm{dy}}{\mathrm{dx}} &= 0, \text{ so}\\ \frac{\mathrm{dy}}{\mathrm{dx}} &= -\frac{3x^2}{2y}.\end{aligned}$$

[3 mark(s)]

ii) $\frac{\mathrm{dy}}{\mathrm{dx}}$ when $3x^2 + xy^2 - 9 = 0$, so we can differentiate both sides of this equation as follows

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dx}}\left[3x^2 + xy^2 - 9\right] &= \frac{\mathrm{d}}{\mathrm{dx}}[0] \\
6x + y^2 + 2xy\frac{\mathrm{dy}}{\mathrm{dx}} &= 0, \text{ so} \\
\frac{\mathrm{dy}}{\mathrm{dx}} &= -\frac{6x + y^2}{2xy}
\end{aligned}$$

[3 mark(s)]

3) If $x = \gamma \sin(\theta)$, and $y = \cos(\theta)$, evaluate $\frac{\mathrm{dy}}{\mathrm{dx}}$, and $\frac{\mathrm{d^2y}}{\mathrm{dx^2}}$.

$$\begin{aligned}
\frac{\mathrm{dy}}{\mathrm{dx}} &= \frac{\mathrm{dy}}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{dx}}, \text{ where} \\
\frac{\mathrm{dy}}{\mathrm{d}\theta} &= -\sin(\theta), \\
\frac{\mathrm{dx}}{\mathrm{d}\theta} &= \gamma\cos(\theta), \text{ so} \\
\frac{\mathrm{dy}}{\mathrm{dx}} &= \frac{-\sin(\theta)}{\gamma\cos(\theta)} \\
&= = -\frac{1}{\gamma}\tan(\theta).
\end{aligned}$$

The second derivative is given by

$$\begin{aligned}
\frac{\mathrm{d^2y}}{\mathrm{dx^2}} &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{\mathrm{dy}}{\mathrm{dx}}\right] \times \frac{\mathrm{d}\theta}{\mathrm{dx}}, \\
&= \frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{-\sin(\theta)}{\gamma\cos(\theta)}\right] \times \frac{1}{\gamma\cos(\theta)}, \\
&= \frac{-1}{\gamma^2}\frac{\left[\cos^2(\theta) + \sin^2(\theta)\right]}{\cos^3(\theta)}, \\
&= \frac{-\sec^3(\theta)}{\gamma^2}.
\end{aligned}$$

[8 mark(s)]

4) Find the angle between $y = x^2$, and $y = x$ at $x = 1$. The gradient of a curve is given by $\frac{\mathrm{dy}}{\mathrm{dx}}$, and the angle θ between the tangent to a curve at some point x, and the x-axis is given by $\arctan\left(\frac{\mathrm{dy}}{\mathrm{dx}}\right)$, so, for $y = x^2$, $y' = 2x$. At $x = 1$, $y' = 2$, and $\theta = \arctan(2) = 63.4°$ (or 1.107 rad). For $y = x$, $y' = 1$, and $\theta = \arctan(1) = 45°$ (or $\pi/4$ rad). So the angle between these two curves is just $\arctan(2) - \arctan(1) = 18.4°$ (or 0.332 rad).

[3 mark(s)]

5) Find the equations of the tangent, and normal to the curve defined by $y = \sin(\theta)$, and $x = \cos(\theta)$ when $\theta = \frac{\pi}{4}$.

$$\begin{aligned}\frac{\mathrm{dy}}{\mathrm{d}\theta} &= \cos(\theta),\\ \frac{\mathrm{dx}}{\mathrm{d}\theta} &= -\sin(\theta), \text{ so}\\ \frac{\mathrm{dy}}{\mathrm{dx}} &= \frac{\mathrm{dy}}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{dx}},\\ &= -\frac{\cos(\theta)}{\sin(\theta)}.\end{aligned}$$

At $\theta = \frac{\pi}{4}$, the gradient is -1, $x = 1/\sqrt{2}$, and $y = 1/\sqrt{2}$. So the tangent to the curve is $y = -x + \sqrt{2}$, and the normal to the curve is $y = x$.

[4 mark(s)]

6)

$$\begin{aligned}\dot{x} &= -\beta e^{-\beta t}\cos(\omega t) - \omega e^{-\beta t}\sin(\omega t)\\ \ddot{x} &= \beta^2 e^{-\beta t}\cos(\omega t) + \omega\beta e^{-\beta t}\sin(\omega t) + \beta\omega e^{-\beta t}\sin(\omega t) - \omega^2 e^{-\beta t}\cos(\omega t)\end{aligned}$$

so,

$$\begin{aligned}\frac{\mathrm{d^2x}}{\mathrm{dt^2}} + A\frac{\mathrm{dx}}{\mathrm{dt}} + Bx &= \beta^2 e^{-\beta t}\cos(\omega t) + 2\omega\beta e^{-\beta t}\sin(\omega t) - \omega^2 e^{-\beta t}\cos(\omega t)\\ &\quad + A\left(-\beta e^{-\beta t}\cos(\omega t) - \omega e^{-\beta t}\sin(\omega t)\right) + B\left(e^{-\beta t}\cos(\omega t)\right)\\ &= e^{-\beta t}\cos(\omega t)\left[\beta^2 - \omega^2 - A\beta + B\right] + e^{-\beta t}\sin(\omega t)\left[2\omega\beta - A\omega\right]\\ &= 0\end{aligned}$$

This is only satisfied for all x if $A = 2\beta$, and $B = \omega^2 + \beta^2$.

[5 mark(s)]

7) Prove that $y = A(\sin(\omega x) + \cos(\omega x))$ satisfies $\frac{\mathrm{d^2y}}{\mathrm{dx^2}} \propto \mathrm{y}$, and determine the constant of proportionality.

$$\begin{aligned} y &= A(\sin(\omega x) + \cos(\omega x)) \\ y' &= \omega A(\cos(\omega x) - \sin(\omega x)) \\ y'' &= -\omega^2 A(\sin(\omega x) + \cos(\omega x)) \end{aligned}$$

so y is proportional to $\frac{\mathrm{d^2y}}{\mathrm{dx^2}}$, and the constant of proportionality is $-\omega^2$.

[4 mark(s)]