1B45 Mathematical Methods Problem Class 2 2005/2006

Week starting Monday 31st. October

1. Differentiation using the product and/or the chain rule.

(a)

$$\text{If } \ y = \left(x + \frac{1}{x}\right)^2 \ \text{ show that } \ \frac{dy}{dx} = 2x - \frac{2}{x^3} \ .$$

(b) Obtain

$$\frac{dy}{dx} \ \text{ if } \ y = \sqrt{x} + \frac{1}{\sqrt{x}} \ .$$

(c)

$$\text{If } \ y = \frac{\sqrt{x}+1}{\sqrt{x}-1} \ \text{ show that } \ \frac{dy}{dx} = \frac{1}{\sqrt{x}} \frac{-1}{(\sqrt{x}-1)^2} \ .$$

(d) Obtain

$$\frac{dy}{dx} \ \text{ if } \ y = sin^{-1} x^2 \ .$$

(e)

$$\text{If } \ y = ln \ \frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}} \ \text{ show that } \ \frac{dy}{dx} = \frac{\sqrt{a}}{\sqrt{x}(a-x)} \ .$$

2.Integration by inspection or by the substitutions given.

(Note the constant of integration is not included in what follows.)

(a)

$$\text{Show that } \ \int \frac{dx}{\sqrt{x}} = 2\sqrt{x} \ .$$

(b) Obtain

$$\int \frac{dx}{\sqrt{ax+b}}$$

(c)

$$\text{Show that } \ \frac{d}{dx} tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{(x^2+a^2)} \ \text{ and hence determine } \ \int \frac{dx}{x^2+a^2}.$$

Repeat the determination of the integral by making the substitution $x = atan\theta$.

(d)

$$\text{Determine } \ \int \frac{dx}{e^x + e^{-x}} \ \text{ by making the substitution } \ u = e^x.$$

You may also use any standard result already obtained in this question.

(e)

$$\text{Evaluate } \ \frac{2\pi\alpha}{s} \int_{-1}^{+1} \frac{dcos\theta}{(1 + cos\theta + \frac{2m_e^2}{s})} \ .$$

3.

$$\text{Show that } \int_0^\infty e^{-\alpha x} dx = \frac{1}{\alpha}.$$

Using integration by parts evaluate the following definite integrals

$$\int_0^\infty x e^{-\alpha x} dx \ , \ \int_0^\infty x^2 e^{-\alpha x} dx \ \text{ and } \ \int_0^\infty x^3 e^{-\alpha x} dx,$$

and show that your results agree with the general formula

$$\int_0^\infty x^n e^{-\alpha x} dx = \frac{n!}{\alpha^{n+1}} \ .$$

Obtain the same result by differentiating both sides of the equation at the start of this question with respect to α.

4. Ice of thickness x metres and density ρ has formed on the surface of a lake. The temperature of the lake's surface at $x = 0$, T_s is lower than that of the water just below the ice, which is at 0^0C. For each kilogram of ice that forms L joules of latent heat Q is released. The rate of heat flow dQ/dt is given by

$$\frac{dQ}{dt} = -\kappa A \frac{dT}{dx}$$

where A is the area through which the heat flows, κ is the coefficient of thermal conductivity of ice and $\frac{dT}{dx}$ is the temperature gradient in the ice, which you can assume to be uniform.

Obtain an equation for the time t, at which a thickness of ice x has formed in terms of x, ρ, κ, L and T_s.