Functions satisfying differential equations

An equation containing derivatives or differentials is called a *differential equation.* A solution of a differential equation is a function which satisfies the equation. Show that the following functions are solutions of the given differential equations:

(1) $y = e^x - 1$; $\frac{dy}{dx} = y + 1$

$$\frac{dy}{dx} = e^x \quad \text{Substituting in the D.E.} \ \frac{dy}{dx} = (e^x - 1) + 1 = e^x.$$

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(2) $y = 3e^{-2x} + x - 1$; $y' + 2y = 2x - 1$.
$y' = -6e^{-2x} + 1$. Therefore we have

$$\begin{aligned} y' + 2y &= -6e^{-2x} + 1 + 2(3e^{-2x} + x - 1) \\ &= -6e^{-2x} + 1 + 6e^{-2x} + 2x - 2 = 2x - 1. \end{aligned}$$

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(3) $y = 3\cos 2x$; $y'' + 4y = 0$.

$$y' = -6\sin 2x, \quad y'' = -12\cos 2x. \quad \text{Therefore we get}$$
$$y'' + 4y = -12\cos 2x + 4(3\cos 2x) = 0.$$

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(4) $y = e^{2t} + e^{-3t}$; $y'' + y' - 6y = 0$.

$$y' = 2e^{2t} - 3e^{-3t}, \quad y'' = 4e^{2t} + 9e^{-3t}.$$

For these more complicated problems it is easier to write the solution is column form.

$$\begin{aligned} y'' &= 4e^{2t} + 9e^{-3t} && (1) \\ y' &= 2e^{2t} - 3e^{-3t} && (2) \\ -6y &= -6e^{2t} - 6e^{-3t} && (3) \end{aligned}$$

(1)+(2)+(3)$\Rightarrow$ $y'' + y' - 6y = (4 + 2 - 6)e^{2t} + (9 - 3 - 6)e^{-3t} = 0$

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(5) $y = te^{3t}$ $y'' - 3y' = 3e^{3t}$.

$$y' = t(3e^{3t}) + e^{3t}, \ y'' = 3t(3e^{3t}) + 3e^{3t} + 3e^{3t} = 9te^{3t} + 6e^{3t}.$$

$$\begin{aligned} y'' &= 9te^{3t} + 6e^{3t} && (4) \\ -3y' &= -9te^{3t} - 3e^{3t} && (5) \end{aligned}$$

(4)+(5)$\Rightarrow$ $y'' - 3y' = (9 - 9)te^{3t} + (6 - 3)e^{3t} = 3e^{3t}$.

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(6) $y = e^{2x}\sin 3x$; $y'' + 2y' + y = 18e^{2x}\cos 3x$.

$$\begin{aligned} y' &= e^{2x}(3\cos 3x) + 2e^{2x}\sin 3x = e^{2x}(3\cos 3x + 2\sin 3x) \\ y'' &= e^{2x}(-9\sin 3x + 6\cos 2x) + 2e^{2x}(3\cos 3x + 2\sin 3x). \end{aligned}$$

$$\begin{aligned} y'' &= e^{2x}(-5\sin 3x + 12\cos 3x) && (6) \\ y' &= e^{2x}(6\cos 3x + 4\sin 3x && (7) \\ y &= e^{2x}\sin 3x && (8) \end{aligned}$$

(6)+(7)+(8)

$$\Rightarrow\ y'' + y' + y = e^{2x}\left\{(-5+4+1)\sin 3x + (12+6)\cos 3x\right\} = 18\cos 3x$$

Assignment for functions satisfying D.E.

(1) $y = 2x^3 - x^2;\ xy' - 3y = x^2.$
(2) $y = \frac{1}{2}\cos x;\ y'' + 9y = 4\cos x.$
(3) $y = \sin x + \cos x - e^{-x};\ y' + y = 2\cos x.$
(4) $y = 3e^{-2x} + 4e^{-x};\ y'' + 3y' + 2y = 0.$
(5) $y = t\sin 2t;\ y'' + 4y = 4\cos 2t.$
(6) $y = 5xe^{-4x};\ y'' + 8y' + 16y = 0.$
(7) $y = e^{-3x}\sin 2x;\ y'' - 5y = -12e^{-3x}\cos 2x.$

Applications of differential equations

(1) When a hockey player strikes a puck with a certain force at $t = 0$ the puck moves along the ice with velocity at any time thereafter given by the expression $v(t) = 27 - 9\sqrt{t}\ m/sec$. How far would the puck travel on a long sheet of ice before coming to rest? (Asumme $s(t) = 0$ at $t = 0$.)
Solution: $\frac{ds}{dt} = v(t)$. Therefore

$$\begin{aligned} s &= \int ds = \int v(t)dt = \int (27 - 9\sqrt{t})dt = 27t - \frac{9t^{3/2}}{3/2} + C \\ &= 27t - 6t^{3/2} + C. \end{aligned}$$

At $t = 0$, $s(t) = 0 = 27(0) - 6(0) + C$. Therefore $C = 0$. When the puck stops, $v(t) = 0 = 27 - 9\sqrt{t}$. Hence $9\sqrt{t} = 27$ so $\sqrt{t} = 3$. Therefore $t = 9\ sec$. Substituting in the equation for s, $s(9) = 27(9) - 6(9)^{3/2} = 81\ m$.

(2) A population of elephants grows ar a rate which is proportional to the square root of the population P. That is $\frac{dP}{dt} = k\sqrt{P}$ where k is a constant ($k = 0.2$ here) and t is the time in years. How many elephants will there be in 40 years if there were 16 when $t = 0$? (These are called the initial conditions.)
Solution: Separating the variables we have $\frac{dP}{\sqrt{P}} = 0.2dt$. Now writing this as an integral we have

$$\int \frac{dP}{\sqrt{P}} = \int P^{-1/2}dP = \int 0.2dt.$$

Performing the integration yields

$$\frac{P^{1/2}}{1/2} = 0.2t + C \tag{9}$$

Now applying the initial conditions to (9), $2\sqrt{16} = 0 + C$ so $C = 8$. Substituting this value in (9) we have $2\sqrt{P} = 0.2t + 8$ or $\sqrt{P} = 0.1t + 4$. Squaring both sides, $P = (0.1t + 4)^2$.When $t = 40$ years, $P = [(0.1)(40) + 4]^2 = 64$ elephants.

(3) Suppose that a motorboat is travelling at $10\ m/sec$. At $t = 0$ the motor is switched off. The deceleration due to water resistance is given by $a = \frac{dv}{dt} = -0.04v^2$.
   (a) Find an expression for the velocity of the boat as a function of time?
   (b) How fast will the boat be travelling after 10 *sec.*?

Solution: (a) Separating the variables we have

$$\frac{dv}{-v^2} = 0.04\ dt \quad \text{or} \quad \int -v^{-2}dv = \int 0.04dt.$$

$\frac{-v^{-1}}{-1} = 0.04t + C$ or simplifying $\frac{1}{v} = 0.04t + C$. Now applying the initial conditions, when $t = 0$, $v = 10\ m/sec.$ so $1/10 = C$. Therefore

$\frac{1}{v} = 0.04t + \frac{1}{10} = \frac{0.4t + 1}{10}$. Inverting the equation $v = \frac{10}{1 + 0.4t}$.

(b) When $t = 10\ sec.$,

$$v = \frac{10}{1 + (0.4)(10)} = \frac{10}{1 + 4} = 2\ m/sec.$$

(4) The volume of water in a cylindrical tank of radius $2\ m$ is given by $v = \pi r^2 h = 4\pi h$ where h is the depth. If the initial volume of water in the tank was $16\pi\ m^3$ it takes $80\ min.$ for the tank to drain completely after a valve in the bottom is opened. The water drains according to Toricelli's Law which states that

$$dV/dt = -k\sqrt{h} \quad (\text{For this tank} k = \frac{\pi}{5}) \tag{10}$$

(a) Find an expression for the volume of the water in the tank as a function of t.
(b) Find V when $t = 40\ min.$

Solution:(a) Since $V = 4\pi h,\ h = \frac{V}{4\pi}$ so that sunstituting in (10) gives

$$\frac{dV}{dt} = -\frac{\pi}{5}\sqrt{\frac{V}{4\pi}} = -\frac{\sqrt{\pi}}{10}\sqrt{V}.$$

Separating the variables yields the equation

$$\int \frac{DV}{\sqrt{V}} = \int -\frac{\sqrt{\pi}}{10}dt \text{ with solution } 2\sqrt{V} = -\frac{\sqrt{\pi}}{10}t + C.$$

When $t = 0$, $V = 16\pi$ so $C = 2\sqrt{16\pi} = 8\sqrt{\pi}$. Hence $\sqrt{V} = -\frac{\sqrt{\pi}}{20}t + 4\sqrt{\pi}$. Factoring out $4\sqrt{\pi}$ we obtain $\sqrt{V} = 4\sqrt{\pi}(1 - \frac{t}{80})$ and squaring both sides

$$V = 16\pi\left(1 - \frac{t}{80}\right)^2.$$

(b) When $t = 40\ min.$, $V = 16\pi\left(1 - \frac{40}{80}\right)^2 = 4\pi\ m^3$ (1/4 of the volume).

Assignment on differential equations

(1) (a) Aspace ship is launched from Earth. If the primary booster rockets provide an accelaration of $a = 3\sqrt{t}\ m/sec.^2$, find the velocity when all the fuel is burned up after 225 $sec.$, and the booster rockets are ejected.
(b) Later in the mission secondary rockets are fired which provide an accelaration of $2t\ m/sec.^2$ If these rockets burn for 40 $sec.$, find the new velocity.

(2) A puck is shot along the surface of a frozen canal. The velocity of the puck is given by the expression $v(t) = 32 - 16t^{1/3}\ m/sec.$ How long will it take for the puck to stop and how far will the puck travel before stopping?

(3) a population of rabbits grows at the rate $\frac{dP}{dt} = 0.2P^{3/4}$. If at $t = 0$ there were 16 rabbits, how many will there be after 20 months? (t is in months)

(4) a train is travelling at 64 $km/hr.$ when the caboose suddenly becomes detached. If the friction of the rails provides a deceleration of $a = \frac{dv}{dt} = -2v^{3/2}\ km/hr.^2$, how fast will the caboose be going after $\frac{1}{4}$ of an hour? (Hint: Solve the D.E. $-\frac{dv}{v^{3/2}} = 2dt$)

(5) A population of whales grows at the rate $\frac{dP}{dt} = \frac{2}{3}P^{1/4}$. If $P = 16$ when $t = 0$ years, how many whales will there be after 38 years?

(6) The volume of water in a tank is given by$V = 8h^{3/2}$. Therefore

$$h^{3/2} = \frac{V}{8} \quad \text{or} \quad h = \left(\frac{V}{8}\right)^{2/3} = \frac{V^{2/3}}{4}.$$

If the water drains from the tank according to Toricelli's law, then

$$\frac{dV}{dt} = -2\sqrt{h} = -2\sqrt{\frac{V^{2/3}}{4}} = -2\frac{V^{1/3}}{2} = -V^{1/3} \Rightarrow \frac{dV}{V^{1/3}} = -dt.$$

(a) If the tank is initially filled with 125 m^3 of water, find an expression for the volume of water in the tank at any time t. (t is in minutes).

(b) what is the volume in the tank after 24 minutes?

(c) How long does it take for the tank to drain completely?

(7) A population of deer grows at the rate $\frac{dP}{dt} = P^{2/3}$ where t is the time in years. If the initial population was 64, how many will there be after 3 years?

(8) The volume of water in a canonical tank in which the radius is $\frac{1}{3\sqrt{\pi}}$ times the height can be expressed as

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{3\sqrt{\pi}}\right)^2 h = \frac{h^3}{27} \Rightarrow h = 3V^{1/3}.$$

The tank is initially filled with 729 m^3 of water which drains from the tank according to Toricelli's law, $\frac{dV}{dt} = -\sqrt{3}\sqrt{h}\ m^3/min.$ Hence

$$\frac{dV}{dt} = -\sqrt{3}\sqrt{3V^{1/3}} = -3V^{1/6}.$$

(a) Find an expression for the volume of the water remaining in the tank after $t = 0$.

(b) Calculate how long it will take for the tank to be empty.

(9) A motorboat is travelling at 40 $km/hr.$ when the engine is switched off. The water provides a deceleration of

$$a = \frac{dv}{dt} = -1.8v^2\ km/hr.^2$$

Find the velocity after $\frac{1}{24}$ $hr.$

Answers: (1) (a) 6750 $m/sec.$ (b) 8350 m/sec. (2) 8 $sec.$, 64 m
(3)81 rabbits. (4)$\frac{69}{4} = 7.1\ km/hr.$ (5) 81 whales
(6) (a)$V = (25 - \frac{2}{3}t)^{3/2}$ (b) 27 m^3 (c) 37.5 $min.$ (7) 125 deer.
(8) (a) $V = (243 - \frac{5}{2}t)^{6/5}$ (b) 97.2 $min.$ (9) 10$km/hr.$