COURSEWORK #5 SOLUTIONS

(1.) $$L = m_s (G M_p a)^{1/2}$$

$$\text{TORQUE} = \bar{\Gamma} = \frac{d\bar{L}}{dt}$$

FOR A PLANAR PROBLEM AND ASSUMING $\bar{L}$ & $\bar{\Gamma}$ ARE PARALLEL

$$\Gamma = \frac{dL}{dt} = \frac{d}{dt} \underbrace{m_s (G M_p)^{1/2}}_{\text{CONSTANTS}} a^{1/2} \quad \checkmark [1/2 \text{ MARK}]$$

$$\Gamma = \frac{m_s (G M_p)^{1/2}}{2 a^{1/2}} \left(\frac{da}{dt}\right) \quad \checkmark [1/2 \text{ MARK}]$$

NOTE: YOU NEED TO USE THE ~~CHAIN~~ RELATION

$$\frac{d}{dt} a^{1/2} = \frac{1}{2}\left(\frac{1}{a^{1/2}}\right) \frac{da}{dt}$$

WHICH COMES FROM THE CHAIN RULE

$$f \to f(a) \qquad \frac{df}{dt} = \left(\frac{df}{da}\right)\left(\frac{da}{dt}\right)$$

(2.) THE TERM $\text{SIGN}\left(\Omega_p - \omega_s\right)$ HAS UNIT LENGTH. THEREFORE IT DOES NOT AFFECT THE MAGNITUDE OF THE TORQUE.

HOWEVER IT DOES DETERMINE THE SIGN OF THE TORQUE

FOR $\Omega_p > \omega_s$ THE PLANET ROTATES FASTER THAN THE SATELLITE ORBIT, THE TORQUE IS POSITIVE AND ADDS '+' ANGULAR MOMENTUM.

- IF $\omega_s > 0$ (CORRESPONDING TO A PROGRADE ORBIT) THE THE SATELLITES ORBITAL ANGULAR MOMENTUM INCREASES IN THE '+' DIRECTION AND ITS ORBIT EXPANDS. [1/2 MARK]
- IF $\omega_s < 0$ (CORRESPONDING TO A RETROGRADE ORBIT) THE SATELLITE'S ORBITAL ANGULAR MOMENTUM IS NEGATIVE AND ADDING 'POSITIVE' ANGULAR

PTO ⟹

ANGULAR MOMENTUM WILL CAUSE THE ORBIT TO CONTRACT

FOR $\Omega_p < \omega_s$ THE TORQUE IS NEGATIVE AND CAUSES THE ANGULAR MOMENTUM OF THE ORBIT TO INCREASE IN THE '−' DIRECTION. 1½ [½ MARK]

EXAMPLES INCLUDE PROGRADE SATELLITES INSIDE SYNCHRONOUS. THE '−' TORQUE CAUSES THE ORBIT TO CONTRACT AND DECAY INTO THE PLANET.

2. (b)

$$\frac{dl}{dt} = \left(\frac{3}{2}\right)\left(\frac{k_2}{Q}\right)\frac{G M_s^2 R_p^5}{r^6}\,\text{SIGN}(\Omega_p - \omega_s) = M_s\left(\frac{GM_p}{a}\right)^{1/2}\frac{da}{dt}$$

WE DROP THIS TERM MINDFUL THAT IT DETERMINES THE SIGN OF THE TORQUE

FOR A CIRCULAR ORBIT $r = a$

$$\frac{3}{2}\left(\frac{k_2}{Q}\right)\frac{G M_s^2 R_p^5}{a^6} = M_s\left(\frac{GM_p}{a}\right)^{1/2}\frac{da}{dt}$$

SEPARATE VARIABLES a & t AND INTEGRATE

$$\text{SIGN}(\Omega_p - \omega_s)\,\frac{3}{2}\left(\frac{k_2}{Q}\right)\left(\frac{G}{M_p}\right)^{1/2} M_s R_p^5 \int_{t_0}^{t_1} dt = \int_{a_0}^{a_1} a^{11/2}\,da$$

[1 MARK]

ALL CONSTANTS

$$\int_{t_0}^{t_1} dt = t_1 - t_0 = \Delta t$$

[1 MARK]

$$\int_{a_0}^{a_1} a^{11/2}\,da = \frac{2}{13}\,a^{13/2}\Big]_{a_0}^{a_1} = \frac{2}{13}\left[a_1^{13/2} - a_0^{13/2}\right]$$

$$a_1^{13/2} = \text{SIGN}(\Omega_p - \omega_s)\left(\frac{39}{4}\right)\left(\frac{k_2}{Q}\right)\left(\frac{G}{M_p}\right)^{1/2} M_s R_p^5\,\Delta t + a_0^{13/2}$$

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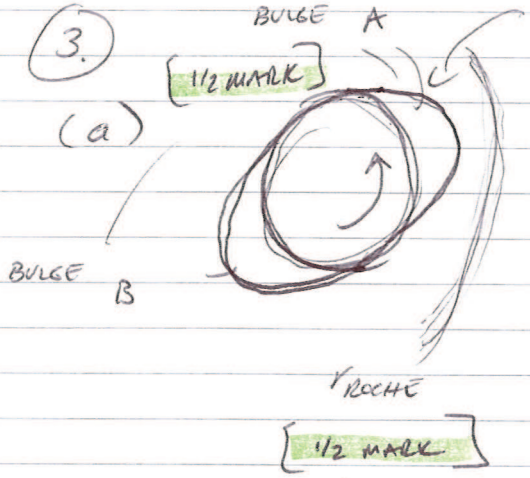
PTO

[1 MARK]

$$a_1 = \left\{ a_0^{13/2} + \mathrm{SIGN}(\Omega_p - \omega_s) \left(\frac{39}{4}\right)\left(\frac{k_2}{Q}\right)\left(\frac{G}{M_p}\right)^{1/2} M_s R_p^{\,5} \Delta t \right\}^{2/13}$$

3.

(a)

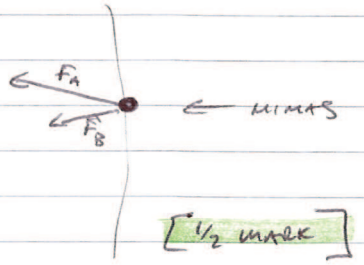


[1/2 MARK]

$\Omega_p > \omega_s$ SO TIDAL BULGES LEAD AHEAD OF MIMAS-SATURN LINE

[1/2 MARK]

SYNCHRONOUS ORBIT
$\sim 1.86\ R_{SATURN}$
FROM EARLIER COURSEWORK



[1/2 MARK]

MIMAS

$a_{MIMAS} = 3.1\ R_S$

r_{ROCHE}

[1/2 MARK]

MIMAS ROCHE RADIUS

$$r_{ROCHE} = \left(\frac{3\rho_{PLANET}}{\rho_{MIMAS}}\right)^{1/3} R_{PLANET}$$

$$\rho_{SATURN} = \frac{M}{VOLUME} = \frac{M}{\frac{4\pi}{3}R^3} = \frac{3M}{4\pi R^3} = \frac{3(5.68\times10^{26}\ kg)}{4\pi(6.033\times10^{7}\ m)^3}$$

$$\rho_{SATURN} = 618\ kg/m^3$$

$$\rho_{MIMAS} = \frac{3M}{4\pi R^3} = \frac{3(3.85\times10^{19}\ kg)}{4\pi(1.99\times10^{5}\ m)^3} = 1170\ kg/m^3$$

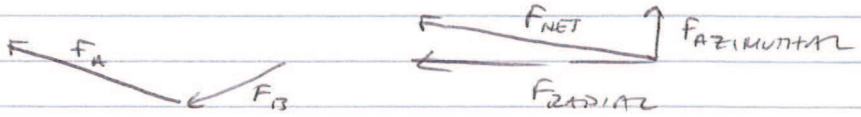
$$r_{ROCHE} = \left(\frac{3(618\ kg/m^3)}{(1170\ kg/m^3)}\right)^{1/3} R_{SATURN}$$

$$r_{ROCHE} = 1.17\ R_{SATURN} \quad \text{OR} \quad 7.03\times10^{7}\ m$$

(b) BECAUSE THE TIDAL BULGE ON THE MIMAS FACING SIDE OF SATURN (BULGE A) IS CLOSER TO MIMAS, THE GRAVITATIONAL FORCE THIS BULGE EXERTS ON MIMAS (F_A) IS LARGER THAN THE FORCE EXERTED ON MIMAS BY THE BULGE ON THE OPPOSITE SIDE (F_B)

$$|\vec{F}_A| > |\vec{F}_B|$$

THE NET FORCE $\vec{F}_A + \vec{F}_B = \vec{F}_{NET}$



IT HAS A SMALL AZIMUTHAL COMPONENT THAT RESULTS IN A TORQUE AND EXPANSION OF MIMAS ORBIT.

[1 MARK]

(c) TO HAVE ALWAYS BEEN TIDALLY EVOLVING TO LARGER SEMI-MAJOR AXIS, MIMAS MUST HAVE BEEN ORBITING ABOVE SATURN'S SYNCHRONOUS ORBIT.

$$a_{MIMAS} > a_{SYNCHRONOUS}$$

$$> 1.86\, R_{SATURN}$$

TO AVOID BEING TIDALLY DISRUPTED MIMAS MUST ALWAYS HAVE BEEN ORBITING SATURN ABOVE THE ROCHE RADIUS

$$a_{MIMAS} > r_{ROCHE}$$

$$> 1.17\, R_{SATURN}$$

IN THIS CASE THE SYNCHRONOUS ORBIT IS MORE RESTRICTIVE ($a_{SYNCHRONOUS} > a_{ROCHE}$)

SO MIMAS ORBIT COULD NOT HAVE BEEN CLOSER TO SATURN THAN THE SYNCHRONOUS ORBIT WHEN IT FORMED

$$MIN(a_{MIMAS}) = a_{SYNCHRONOUS} = 1.86 R_{SATURN}$$

[1 MARK]

(d) SATURN AND MIMAS PROBABLY FORMED ABOUT 4.5×10^9 YEARS AGO.

$$\Delta t = 4.5 \times 10^9 \text{ yr}$$

$$= 4.5 \times 10^9 \text{ yr} \left(\frac{365.25 \text{ d}}{1 \text{ year}}\right)\left(\frac{24 \text{ hrs}}{1 \text{ d}}\right)\left(\frac{3600 \text{ s}}{1 \text{ hr}}\right)$$

$$\Delta t = 1.42 \times 10^{17} \text{ s}$$

$$a_{SYNCH} = 1.12 \times 10^8 \text{ m} \qquad a_{MIMAS} = 1.85520 \times 10^8 \text{ m}$$

$$\left(\frac{K_2}{Q}\right) \text{SIGN}(\Omega_p - \omega_s) \frac{3}{2} \left(\frac{G}{M_p}\right)^{1/2} M_S R_p^5 \Delta t = \frac{2}{13}\left[a_{MIMAS}^{13/2} - a_{SYNCH}^{13/2}\right]$$

M_S: SATURN'S MASS; M_p: MIMAS MASS; R_p: SATURN'S RADIUS

SOLVING FOR $\left(\frac{K_2}{Q}\right)$

$$\left(\frac{K}{Q}\right) = + \left(\frac{4}{39}\right)\left(\frac{M_{SATURN}}{G}\right)^{1/2}\left(\frac{1}{M_{MIMAS} R_{SATURN}^5 \Delta t}\right)\left(a_{MIMAS}^{13/2} - a_{SYNCH}^{13/2}\right)$$

$$= \left(\frac{4}{39}\right)\left(\frac{5.68 \times 10^{26} \text{ kg kg}^2}{6.67 \times 10^{-11} \text{ N·m}^2}\right)^{1/2}\left[\frac{1}{(3.85 \times 10^{17} \text{ kg})(6.033 \times 10^7 \text{ m})^5 (1.42 \times 10^{17} \text{ s})}\right]$$

$$\times \left\{(1.855 \times 10^8 \text{ m})^{13/2} - (1.12 \times 10^8 \text{ m})^{13/2}\right\}$$

$$\left(\frac{K}{Q}\right) = 3.67 \times 10^{-5}$$

[1 MARK]

IF SATURN IS A FLUID PLANET WITH NO STRENGTH THEN THE LOVE NUMBER

$$K_2 = 3/2$$

AND

$$Q_{SATURN} \simeq 4.1 \times 10^4$$ NOT VERY DISSIPATIVE OR OF HIGH 'QUALITY'

RECALL THE DEFINITION OF THE QUALITY FACTOR

$$Q = \frac{2\pi E_0}{\Delta E}$$

E_0 ← PEAK ENERGY STORED IN SYSTEM

ΔE ← ENERGY DISSIPATED IN EXACT CYCLE (OR ORBIT IN THIS CASE)

4.

(a) HERE WE USE THE EXPRESSION DERIVED EARLIER AND THE VALUE OF $\left(\frac{K_2}{Q}\right)$

$$\text{SIGN}(\Omega - \omega)\frac{3}{2}\left(\frac{K_2}{Q}\right)\left(\frac{G}{M_p}\right)^{1/2} M_S R_p^5 \Delta t = \frac{2}{13}\left[a_1^{13/2} - a_0^{13/2}\right]$$

THE SATELLITE IS INSIDE SYNCHRONOUS, SO

$$\text{SIGN}(\Omega - \omega) \Rightarrow -1$$

SUBSTITUTING INTO THE ABOVE EXPRESSION WE GET

$$-\frac{3}{2}\left(\frac{K_2}{Q}\right)\left(\frac{G}{M_p}\right)^{1/2} M_S R_p^5 \Delta t = \frac{2}{13}\left[a_1^{13/2} - a_0^{13/2}\right]$$

AND $a_1 = R_{SATURN}$ - SATELLITE ENDS UP IN SATURN

$a_0 = a_{SYNCHRONOUS}$ - SATELLITE STARTS AT SYNCHRONOUS

SOLVING FOR THE TIME INTERVAL WE GET

$$* \quad \Delta t = \left(\frac{4}{39}\right)\left(\frac{M_{SATURN}}{G}\right)^{1/2}\left(\frac{1}{M_{MIMAS} R_{SATURN}^5}\right)\left(\frac{Q}{K_2}\right)\left[a_0^{13/2} - a_1^{13/2}\right]$$

(NOTE: WE USE THIS EXPRESSION AGAIN LATER.

NOTE $a_0 = a_{SYNCHRONOUS} > a_1 = R_{SATURN}$ SO THIS TERM IS POSITIVE, AS IT SHOULD BE.

$$\Delta t = \left(\frac{4}{39}\right)\left(\frac{5.98\times10^{20}\,kg\cdot kg^2}{6.67\times10^{-11}\,N\cdot m^2}\right)^{1/2}\left(\frac{1}{(3.85\times10^{19}\,kg)(6.033\times10^7\,m)^5}\right)$$

$$\times\left(\frac{1}{3.67\times10^{-5}}\right)\times\left\{(1.12\times10^8\,m)^{13/2} - (6.033\times10^7\,m)^{13/2}\right\}$$

(4.) (i)

$\Delta t = 5.44 \times 10^{15}\ s = 1.72 \times 10^{8}$ YEARS

OR ABOUT 3.8% OF THE AGE OF THE SOLAR SYSTEM.

[1 MARK]

FOR ANY OF THESE

(ii) IF THE SATELLITE WAS 4x MORE MASSIVE

$$\Delta t \propto \frac{1}{M_{SATELLITE}}$$

SO IT WOULD REQUIRE 1/4 AS MUCH TIME TO REACH SATURN'S SURFACE.

[1 MARK]

(6.) AS THE ORBIT OF THE SATELLITE DECAYED TOWARDS THE PLANET AND FOUND ~~INSIDE~~ ITSELF INSIDE ITS ROCHE RADIUS (SEE #3) TIDAL FORCES WOULD BEGIN TO PULL MASS FROM THE OUTER SHELLS OF THE SATELLITE AND IT WOULD ~~SIFE~~ BEGIN SHEDDING MASS.

$r_{ROCHE} = 1.17 R_{SATURN}$ FOR ICY BODIES

THIS TIDAL DISRUPTION COULD PROVIDE THE MASS FOR THE RING SYSTEM.

IF THE RIGIDITY OF THE SATELLITE WERE SUFFICIENTLY HIGH OR THE SATELLITE WAS SUFFICIENTLY DENSE THAT THE ROCHE RADIUS WAS ~~FOUR~~ $r_{ROCHE} < R_{SATURN} \Rightarrow$

(b.) THEN THE SATELLITE MAY HAVE MADE IT TO THE SURFACE OF SATURN. THE RIGIDITY & DENSITIES REQUIRED ARE ABNORMALLY LARGE AND THIS POSSIBILITY IS CONSIDERED UNLIKELY.

[1/2 MARK]

(c.) IF THE RINGS FORMED VIA THIS MECHANISM, IT MUST HAVE OCCURRED EARLY IN THE SOLAR SYSTEM'S HISTORY. $\Longrightarrow$ WITHIN THE FIRST $\hat{\approx} 1.7 \times 10^8$ YEARS

[1/2 MARK]