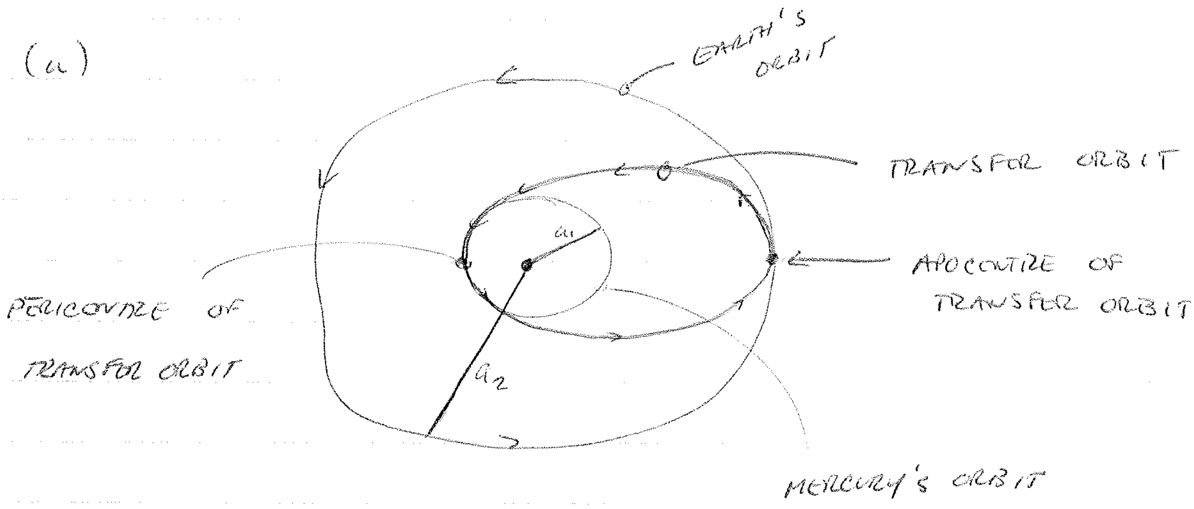
COURSEWORK #3 SOLUTIONS

(1) (a)



[2 MARKS.]

(b) LABEL THE TRANSFER ORBIT WITH 'T' SUBSCRIPTS

PERICENTRE OF TRANSFER ORBIT

$$q_T = a_1$$

$$q_T = a_T(1-e_T) = a_1 \qquad (1)$$

APOCENTRE OF TRANSFER ORBIT

$$Q_T = a_2 = a_T(1+e_T) \qquad (2)$$

[1 MARK]

SOLVING (1) AND (2) FOR a_T & e_T GIVES

$$a_T = \frac{Q_T + q_T}{2} = \frac{0.3871\,\text{AU} + 1.000\,\text{AU}}{2}$$

$$a_T = 0.6936\,\text{AU} \qquad [1 \text{ MARK}]$$

$$e_T = \left(\frac{Q_T}{a_T} - 1\right)$$ ARE MANY WAYS TO SOLVE FOR THIS

$$e_T = \left(\frac{1\,\text{AU}}{0.6936\,\text{AU}} - 1\right)$$

$$e_T = 0.442 \qquad [1 \text{ MARK}]$$

(c) $v = (GM_\odot)^{1/2}\left(\frac{2}{r} - \frac{1}{a}\right)^{1/2}$ [1 MARK]

AT APOCENTRE $r = Q_T = a_2$

$$v_Q = (GM_\odot)^{1/2}\left(\frac{2}{a_2} - \frac{1}{a_T}\right)^{1/2}$$

* CAREFUL TO CONVERT ALL TO METRES

$$= \left\{\left(6.67\times10^{-11}\,\frac{N\cdot m^2}{kg^2}\right)\left(1.98\times10^{30}\,kg\right)\left(\frac{2}{1\,AU} - \frac{1}{0.6936\,AU}\right) \times \left(\frac{1\,AU}{1.496\times10^{11}\,m}\right)\right\}^{1/2}$$

$v_Q = 2.22\times10^{4}\,m/s = 22.1\,km/s$ [1 MARK]

(d) $v_c = \left(\frac{GM_\odot}{a_2}\right)^{1/2} = \left(\frac{\left(6.67\times10^{-11}\,\frac{N\cdot m^2}{kg^2}\right)\left(1.98\times10^{30}\,kg\right)}{\left(1.496\times10^{11}\,m\right)}\right)^{1/2}$

$v_c = 2.94\times10^{4}\,m/s \Rightarrow 29.4\,km/s$

SO THE CHANGE IN SPEED NEEDED IS

$\Delta v = v_f - v_0 = 2.22\times10^{4}\,m/s - 2.94\times10^{4}\,m/s$

$\Delta v = -7.20\times10^{4}\,m/s$ OR $-7.20\,km/s$ [1 MARK]

THE BODY MUST SLOW DOWN.

(e) VELOCITY AT PERICENTRE OF THE TRANSFER ORBIT

$$V_q = (GM)^{1/2}\left(\frac{2}{q_T} - \frac{1}{a_T}\right)^{1/2}$$

$$= \left\{\frac{(6.67\times10^{-11}\,\frac{N\cdot m^2}{kg^2})(1.98\times10^{30}\,kg)}{(1.496\times10^{11}\,m/AU)}\right\}^{1/2}\left(\frac{2}{0.3871\,AU} - \frac{1}{0.6936\,AU}\right)^{1/2}$$

$$= \underline{5.73\times10^{4}\,m/s} \quad = 57.3\ km/s \qquad [1\ \text{MARK}]$$

(f) THE CIRCULAR ORBIT VELOCITY AT MERCURY (a_1) IS

$$V_c = \left(\frac{GM_\odot}{a_1}\right)^{1/2} = \left(\frac{(6.67\times10^{-11}\,\frac{N\cdot m^2}{kg^2})(1.98\times10^{30}\,kg)}{(0.3871\,AU)(1.496\times10^{11}\,m)}\right)^{1/2}$$

$$V_c = 4.78\times10^{4}\,m/s \quad = 47.8\ km/s$$

SO THE SPACECRAFT APPROACHES MERCURY'S ORBIT GOING

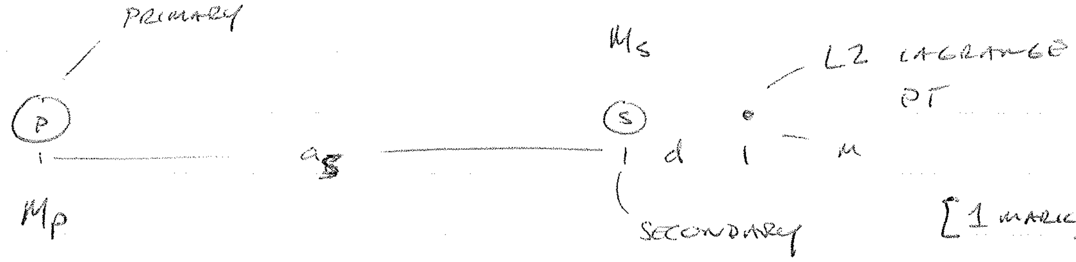
$$\Delta v = \underline{57.3} - \underline{47.8\ km/s} = 9.5\ km/s$$

FASTER THAN THE LOCAL CIRCULAR ORBIT.

[1 MARK]

→ IT WILL TAKE A LOT OF SLOWING DOWN TO GET INTO ORBIT ABOUT THE SUN SIMILAR TO MERCURY'S ⇒ LET ALONE GOING INTO ORBIT ABOUT MERCURY...

2.



[1 MARK]

IN A FRAME ROTATING WITH THE ANGULAR VELOCITY OF THE SECONDARY

$$\omega = \left(\frac{G(M_p + M_s)}{a_s^3} \right)^{1/2}$$

NEWTON'S 2ND LAW FOR CIRCULAR MOTION GIVES

$$F_c = F_{gp} + F_{gs}$$ [2 MARK]

AND THE RADIUS OF CIRCULAR MOTION @ L2 IS

$$r = a_s + d$$ [1 MARK]

(A) $$m(a_s + d)\,\omega^2 = \frac{GM_p m}{(a+d)^2} + \frac{GM_s m}{d^2}$$

(CANCEL MASSES OR JUST WORK WITH ACCELERATION)

NOW ASSUMING $M_p \ll M_\odot$ AND BY IMPLICATION $d \ll a_s$

$$\omega^2 \approx \frac{GM_p}{a_s^3} \qquad (a_s + d)^{-2} \approx \frac{1}{a_s^2}\left(1 - 2\frac{d}{a_s}\right)$$ [2 MARK]

FROM BINOMIAL EXPANSION IN $(1+x)^n \approx 1 + nx$

AND THE EXPRESSION BECOMES

(A) BECOMES

$$a_s \frac{GM_p}{a_s^3} + \frac{GM_p\, d}{a_s^3} = \frac{GM_p}{a_s^2}\left(1 - \frac{2d}{a_s}\right) + \frac{GM_s}{d^2}$$

TERMS CANCEL LEAVING $\Longrightarrow$

$$\frac{\cancel{G}M_p d}{a_s^3} = -\frac{2\cancel{G}M_p d}{a_s^3} + \frac{\cancel{G}M_s}{d^2}$$

$$\frac{3M_p d}{a_s^3} = \frac{M_s}{d^2}$$

SECONDARY MASS

OR $$d = a_s\left(\frac{M_s}{3M_p}\right)^{1/3}$$ [2 MARK]

PRIMARY MASS

SEMI MAJOR AXIS

(b) TO THE LEVEL OF THIS APPROXIMATION THIS [1 MARK] DISTANCE IS EQUIVALENT TO THAT CALCULATED FOR THE L1. [HOWEVER HIGHER ORDER TERMS MAKE SOME DIFFERENCE]

THIS IS AGAIN JUST HILL'S RADIUS.

(3.) (a) THE HILL SPHERE LENGTH GIVES THE APPROXIMATE DISTANCE BETWEEN A SECONDARY (E.G. A PLANET IN THE STAR-PLANET CASE OR A SATELLITE IN THE PLANET-SATELLITE CASE) WHERE THE FORCE FROM THE PRIMARY AND THE SECONDARY JUST BALANCE WITH THE CENTRIFUGAL FORCE.

[2 MARKS]

MATERIAL OR SATELLITES MAY REMAIN ORBITALLY BOUND TO THE SECONDARY IF THEY RESIDE WITHIN THE HILL SPHERE AND THEIR VELOCITY IS LOW ENOUGH.

OUTSIDE THE HILL SPHERE THE FORCE FROM THE PRIMARY AND THE CENTRIFUGAL FORCE KEEP MATERIAL FROM BEING BOUND TO THE SECONDARY

$$r_{HILL} = a\left(\frac{M_s}{3M_p}\right)^{1/3}$$

(b) $$r_{HILL} = r\left(\frac{M_\odot}{3M_{GALAXY}}\right)^{1/3}$$

IN THIS CASE THE SUN IS THE SECONDARY AND THE GALAXY IS THE PRIMARY

DISTANCE OF SUN FROM GALACTIC CENTRE

$$r = 26{,}000\ ly\ (\text{LIGHT YEARS})\left(\frac{9.461\times10^{15}\,m}{1\,ly}\right)$$

$$= 2.46\times10^{20}\,m$$

$$r = 1.64\times10^{9}\,AU$$

$$M_{GALAXY} = 5.8\times10^{11}\,M_\odot$$

(b) CONT

$$r_{HILL} = 1.64 \times 10^{9}\,AU \left(\frac{1\,M_\odot}{3\,(5.8 \times 10^{11}\,M_\odot)} \right)^{1/3}$$

$$\boxed{r_{HILL} = 1.4 \times 10^{5}\,AU}$$ — WITH RESPECT TO THE GALACTIC CENTER

[2 MARKS]

(c.) $$r_{HILL} = r \left(\frac{M_\odot}{3\,M_{\alpha\,CENTAURI}} \right)^{1/3}$$

r — DISTANCE FROM SUN TO α CENTAURI

$$r = (4.37\,ly)\,\frac{(9.461 \times 10^{15}\,m/ly)}{(1.496 \times 10^{11}\,m/AU)}$$

$$r = 2.76 \times 10^{5}\,AU \qquad M_{\alpha\,CENT} = 2\,M_\odot$$

$$r_{HILL} = 2.76 \times 10^{5}\,AU \left(\frac{M_\odot}{3 \cdot 2 \cdot M_\odot} \right)^{1/3}$$

[2 MARKS]

$$\boxed{r_{HILL} = 1.52 \times 10^{5}\,AU}$$ WITH RESPECT TO α · CENTAURI

(d) THE HILL SPHERE OF THE SOLAR SYSTEM IS SMALLER DUE TO THE GALAXY. SO THE GALAXY IS MORE IMPORTANT TO CONSTRAINING THE SIZE OF THE PRESENT DAY SOLAR SYSTEM. HOWEVER THE INFLUENCE OF α CENTAURI IS NOT NEGLIGIBLY WEAK. [2 MARKS]

(e) THE SIZE OF THE OORT COMET CLOUD IS ROUGHLY ~ 50,000 AU OR NEARLY 1 LIGHT·YEAR IN RADIUS. SO THE SIZE OF THE OORT CLOUD IS ALSO ABOUT ~ 0.36 r_{HILL} OR ~ 1/3 OF THE HILL RADIUS. THIS SCALING IS SIMILAR TO THE IRREGULAR SATELLITES OF JUPITER & SATURN. [2 MARKS]

(4.) (a)

$$r_{HILL} = a_{PHOBOS}\left(\frac{M_{PHOBOS}}{3M_{MARS}}\right)^{1/3}$$

$$= 9.38\times10^{6}\,m\left(\frac{1.08\times10^{16}\,kg}{3(6.41\times10^{23}\,kg)}\right)^{1/3}$$

$$r_{HILL} = 1.67\times10^{4}\,m = 16.7\,km$$ — HILL RADIUS OF PHOBOS

$$\left(\frac{r_{HILL}}{R_{PHOBOS}}\right) = \left(\frac{16.7\,km}{11.2\,km}\right) = 1.49$$ [2 MARKS]

(b.) $$r_{ROCHE} = \left(\frac{3\rho_{PRIMARY}}{\rho_{SECONDARY}}\right)^{1/3} R_{PRIMARY}$$

MARS DENSITY IS $= \dfrac{M}{\frac{4\pi}{3}R^3} = \dfrac{(6.41\times10^{23}\,kg)}{\left(\frac{4\pi}{3}\right)(3.394\times10^{6}\,m)^3}$

$$\boxed{\rho_{MARS} = 3910\ kg/m^3}$$

$$r_{ROCHE} = \left(\frac{3(3910\ kg/m^3)}{(1900\ kg/m^3)}\right)^{1/3} R_{MARS}$$

$$\boxed{\begin{aligned} &= 1.83\,R_{MARS} \\ r_{ROCHE} &= 6.21\times10^{6}\,m \end{aligned}}$$ {2 MARKS}

(c) $$r_{HILL} = r\left(\frac{M_s}{3M_p}\right)^{1/3}.$$

THE MASS OF AN $R = 10\,m$ ICY BODY IS

$$M = \left(\frac{4\pi}{3}\right)(1000\ kg/m^3)(10\,m)^3$$

$$M = 4.19\times10^{6}\,kg$$

$$r = 1.2\, R_{SATURN} = 1.2\,(6.033 \times 10^{7} m)$$
$$= 7.24 \times 10^{7} m$$

$$r_{HILL} = (7.24 \times 10^{7} m)\left(\frac{4.19 \times 10^{6} kg}{3(5.68 \times 10^{26} kg)}\right)^{1/3}$$

$$\underline{r_{HILL} = 9.76\ m}$$

[2 MARKS]

ABOUT THE SAME SIZE AS THE PARTICLE

(d.) WHEN IS $R_J = r_{HILL}$?

$$R_J = a_J \left(\frac{M_J}{3M_\odot}\right)^{1/3}$$

$$a_J = R_J \left(\frac{3M_\odot}{M_J}\right)^{1/3}$$

NOTE THIS IS JUST THE DEFINITION OF THE ROCHE RADIUS

$$a_J = R_J \left(\frac{3M_\odot}{M_J}\right)^{1/3}$$

$$a_J = \left(\frac{3\rho_\odot}{\rho_J}\right)^{1/3} R_\odot$$

$$M_J = \frac{4\pi}{3} \rho_J R_J^3$$

$$\rho_J = 1.326 \times 10^{3} kg/m^3$$
$$\rho_\odot = 1.408 \times 10^{3} kg/m^3$$
$$R_\odot = 6.96 \times 10^{8} m = 0.00465\ AU$$

$$a_J = \left[\frac{3(1.408 \times 10^{3} kg/m^3)}{(1.326 \times 10^{3} kg/m^3)}\right]^{1/3} 0.00465\ AU$$

$$\boxed{a_J = 0.00684\ AU = 1.47\ R_\odot}$$

[2 MARKS]

$$\underline{= 1.02 \times 10^{9} m}$$

THE ORBITAL PERIOD HERE IS $P = a^{3/2} = (0.00684)^{3/2}$ YR

$= 5.6 \times 10^{-4}$ YR $= \underline{84.95\ \text{HOURS}}$

5. (a)

$$r_{Hill} = a_J \left(\frac{M_J}{3M_\odot}\right)^{1/3}$$

$$\frac{r_{Hill}}{a_J} = \left(\frac{M_J}{3M_\odot}\right)^{1/3} = \left(\frac{1.90\times10^{27}\,kg}{3\,(1.98\times10^{30}\,kg)}\right)^{1/3}$$

$$\frac{r_{Hill}}{a_J} = 0.0684$$

$$r_{Hill} = 0.356\ AU$$

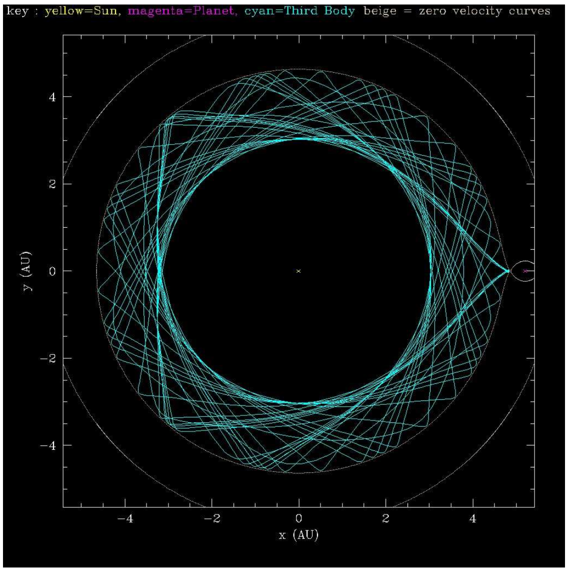
[2 marks]

PHY241 Planetary Systems - Coursework #3

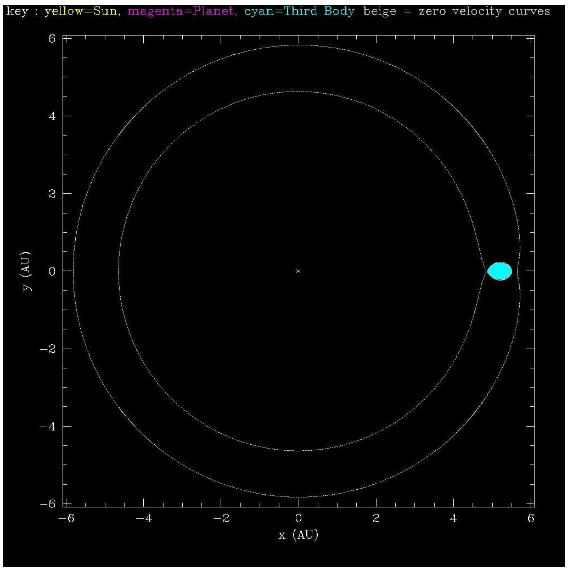
Solutions

Problem 5

Dynamics near L1, L2 , L4 and L5 Points

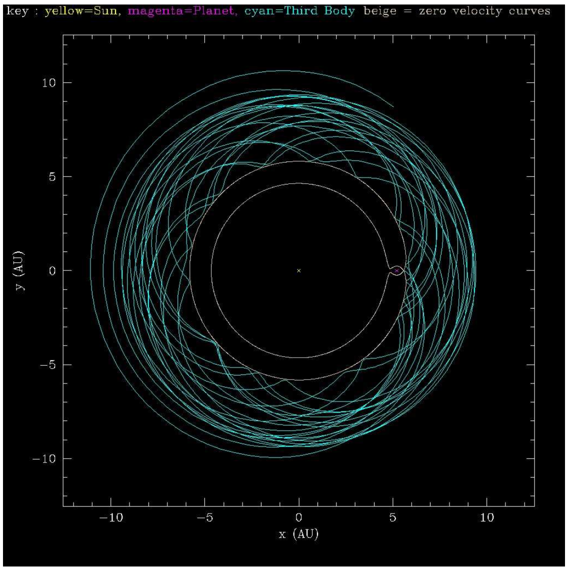


(a) $x = -0.02AU$ interior to the L1 Lagrange Point

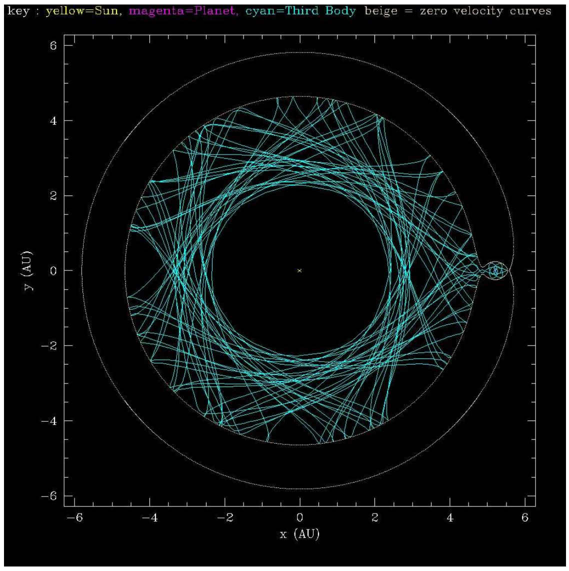


(b) $x = +0.02AU$ interior to the L1 Lagrange Point

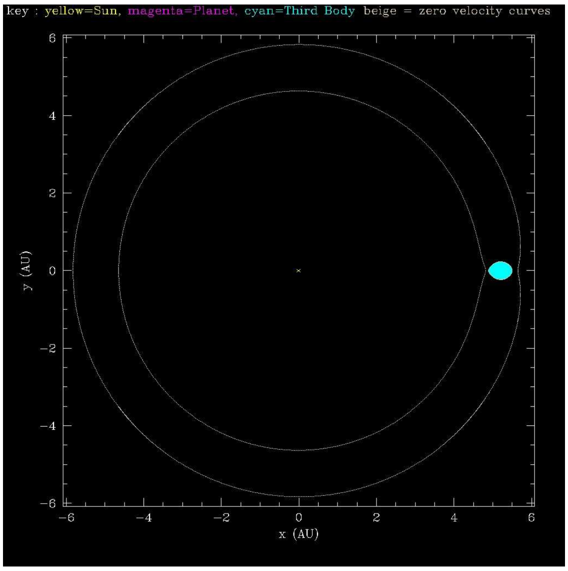
FIGURE 1: The L1 Lagrange Point - In the left frame the particle remains interior to the planet through the simulation and 'reflects' off of the grey 'zero-velocity' curve. For $x = +0.02$AU on the right the particle remains in the Hill sphere of the planet throughout the simulation again 'reflecting' off of the grey 'zero-velocity' curve surrounding the Hill sphere. That the particle does not oscillate about the L1 in either simulation suggests that it is an unstable equilibrium. [2 marks]



(a) $x = +0.02AU$ exterior to the L2 Lagrange Point

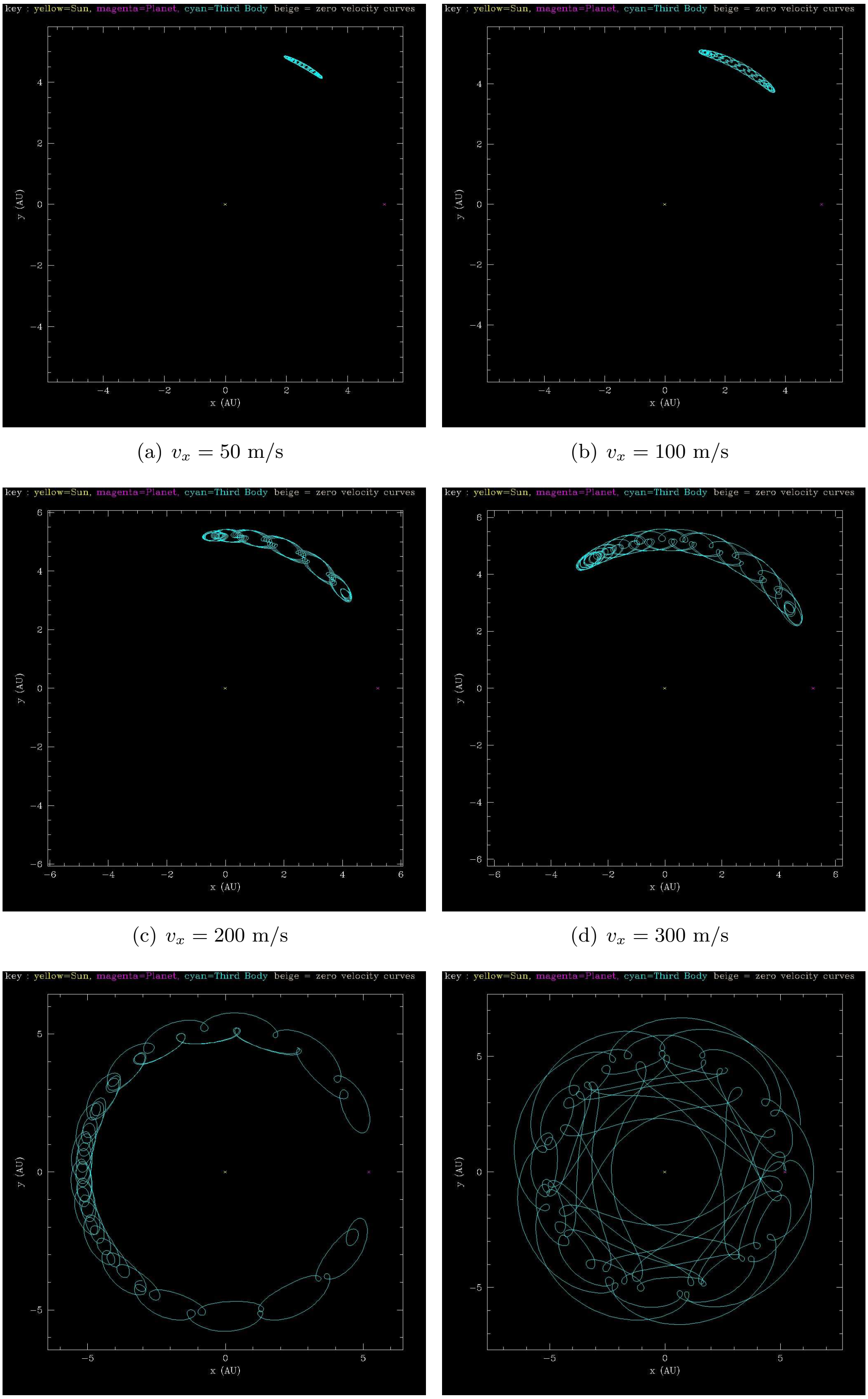


(b) $x = -0.02AU$ interior to the L2 Lagrange Point



(c) $x = -0.07AU$ interior to the L2 Lagrange Point

FIGURE 2: L2 Lagrange Point - In (a) the particle remains exterior to the Hill sphere of the planet throughout the simulation. In (b) The particle starts interior to the planet, but exits the Hill sphere through the L1 and proceeds to travel through the region interior to the L1. In (c) finally the particle remains inside the Hill sphere. In all of these cases the particle again leaves the vicinity of the L2 rather than oscillating about this point, suggesting that this is not a stable equilibrium point.



(a) $v_x = 50$ m/s

(b) $v_x = 100$ m/s

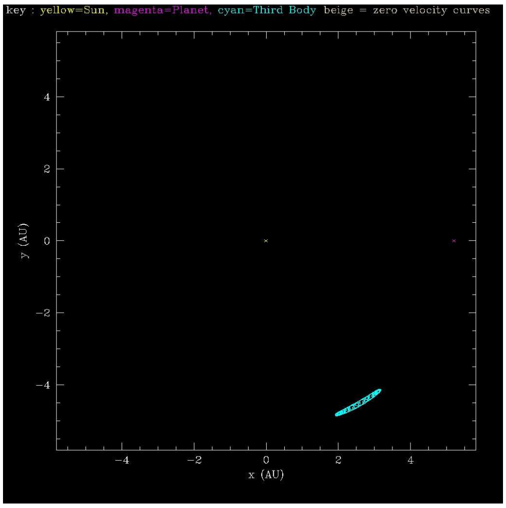
(c) $v_x = 200$ m/s

(d) $v_x = 300$ m/s

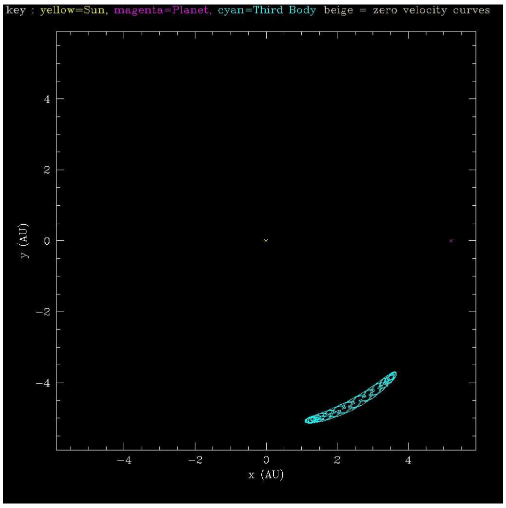
(e) $v_x = 400$ m/s

(f) $v_x = 500$ m/s

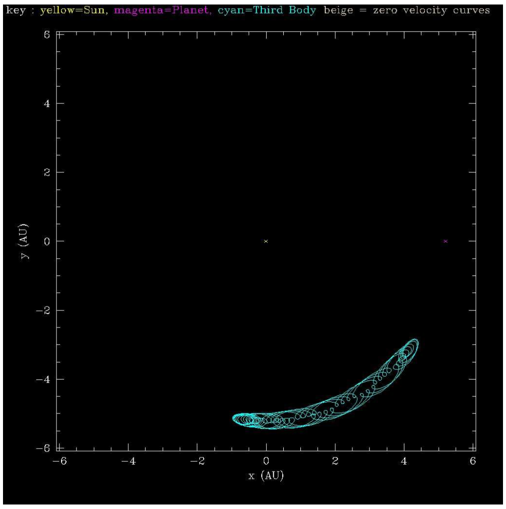
FIGURE 3: L4 Stability - Each figure used a different starting velocity. As the starting velocity is increased the angle of libration about the L4 increases. For very small initial velocity (as in [a]) the particle oscillates about the L4 suggesting it is a stable equilibrium. Note also in [e] that the particle seems to now oscillate about both the L4 and L5 points. This is known as a 'horseshoe' orbit. Saturn's satellites Janus and Epimetheus execute this sort of orbital behavior. Note the different scale in [f] as the particle traverses a much larger region. [2 marks]



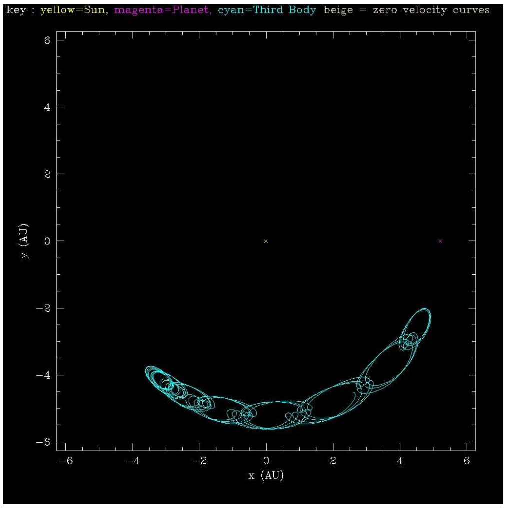
(a) $v_x = 50$ m/s



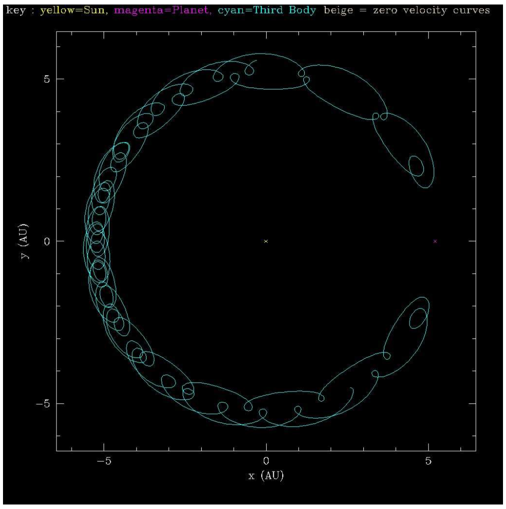
(b) $v_x = 100$ m/s



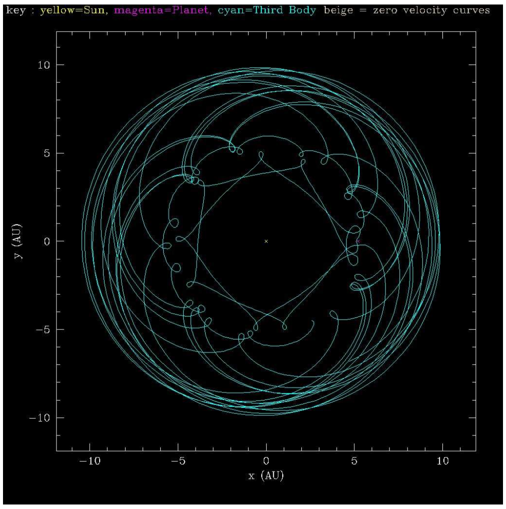
(c) $v_x = 200$ m/s



(d) $v_x = 300$ m/s



(e) $v_x = 400$ m/s



(f) $v_x = 500$ m/s

FIGURE 4: L5 Stability - Each figure used a different starting velocity. Note the similarity to the L4 behaviour. Both appear stable to small perturbations from the equilibrium points, with smaller initial velocities showing smaller amplitude oscillation about the equilibrium point (see e.g.[a]).