PHY218 : Homework 9

1. A particle moving in three dimensions is described by three time-dependent coordinates $(x_1(t), x_2(t), x_3(t))$. By extremizing the integral

$$I = \int_{t_1}^{t_2} L(x_1(t), x_2(t), x_3(t), \dot{x}_1(t), \dot{x}_2(t), \dot{x}_3(t), t)$$

subject to $\delta x_i(t_1) = 0, \delta x_i(t_2) = 0$, derive the Euler-Lagrange equations

$$\frac{\partial L}{\partial x_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i}$$

[8]

2. The Lagrangian for a relativistic particle is

$$L = -m_0 c^2 \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} - V(\mathbf{x})$$

where c is the speed of light, $\mathbf{x} = (x_1(t), x_2(t), x_3(t))$ is the three dimensional coordinate, $V(\mathbf{x})$ is the potential and

$$\mathbf{v} = (\dot{x}_1(t), \dot{x}_2(t), \dot{x}_3(t))$$
$$\mathbf{v}^2 = \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2$$

derive the equation of motion for a relativistic particle using the Euler-Lagrange equations in question 1. Explain how the Newtonian limit is reached. [8+4]

3. A soap film is created between two parallel and co-axial circular wires of radius R, located at $x = \pm x_0$. (A useful diagram is in question B3 of sample paper on the web-site) The surface area of the soap film is a surface of revolution obtained by revolving the curve $y(x)$ about the x-axis by 360 degrees, between the points $x = \pm x_0$.

Show that the area of the soap film is

$$\int_{-x_0}^{x_0} 2\pi y(x)\sqrt{1 + (\frac{dy}{dx})^2}dx \tag{1}$$

Show that this is minimized by the function $y(x) = c\ Cosh\frac{x}{c}$ where c is a constant. Obtain the equation that determines c. [6+8]

4. The Lagrangian for a particle with charge q in an electromagnetic field is described by a scalar potential ϕ and vector potential $\mathbf{A}$ is

$$L = \frac{1}{2}m\mathbf{v}^2 - q\phi + q\mathbf{A} \cdot \mathbf{v}$$

Show that the equation of motion of the particle is

$$m\ddot{\mathbf{x}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

You may use

$$\begin{aligned} E_i &= -\frac{\partial \phi}{\partial x_i} - \frac{\partial A_i}{\partial t} \\ B_i &= (\nabla \times A)_i \end{aligned}$$

You will find it useful to note

$$\frac{d}{dt} A_i(\mathbf{x}(t), t) = \frac{\partial A_i}{\partial t} + \sum_j \dot{x}_j \frac{\partial A_i}{\partial x_j}$$

[15]