PHY218 : Homework 7

1. Given a differential equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = F(x)$$

the Green's Function is required to satisfy

$$[\frac{d^2y}{dx^2} + P(x)\frac{d}{dx} + Q(x)]G(x,x') = \delta(x-x')$$

Show that

$$y(x) = \int G(x,x')F(x')dx'$$

is a particular integral.

[8]

2. The Dirac Delta function obeys the property that

$$\int_{-\infty}^{\infty} F(x)\delta(x-x')dx = F(x')$$

Show that

$$\begin{aligned}
\delta(x) &= \delta(-x) \\
\delta(ax) &= \frac{1}{|a|}\delta(x) \text{ for real } a \neq 0 \\
\int_{-\infty}^{\infty} F(x)\left(\frac{d}{dx}\delta(x-x_0)\right)\, dx &= -\left[\frac{d}{dx}F(x)\right]_{x=x_0} = -F'(x_0) \\
x\delta(x) &= 0 \\
x\delta'(x) &= -\delta(x)
\end{aligned}$$

[4+4+4+5+5]

3. Consider the differential equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = F(x)$$

for y in the interval $[a,b]$, with boundary conditions $y(a) = y(b) = 0$. In class we derived the Green's functions

$$G(x,x') = \frac{y_2(x')y_1(x)}{W(x')} \qquad \text{for} \quad x < x'$$

$$G(x,x') = \frac{y_1(x')y_2(x)}{W(x')} \qquad \text{for } x > x'$$

which obeys

$$[\frac{d^2}{dx^2} + P(x)\frac{d}{dx} + Q(x)]G(x,x') = \delta(x-x') \tag{1}$$

where $W(x') = y_1(x')y_2'(x') - y_2(x')y_1'(x')$.

Calculate the discontinuity in the derivative of the Green's function at $x = x'$ and show how this discontinuity is required by the defining equation (1). Show that the Green's function obeys the boundary conditions

$$G(a,x') = G(b,x') = 0 \text{ for all } x' \tag{2}$$

Use the Green's function (and the result from Question 1) to show that

$$y(x) = y_2(x)\int_a^x \frac{y_1(x')F(x')}{W(x')}dx' + y_1(x)\int_x^b \frac{y_2(x')F(x')}{W(x')}dx' \tag{3}$$

solves the differential equation with the given boundary condition, where $y_1(x), y_2(x)$ are solutions of the homogeneous equation obeying $y_1(a) = y_2(b) = 0$. [4+4+4]

4. Use the result (3) to show that the solution of

$$\frac{d^2y}{dx^2} + y = \mathrm{cosec}x$$

for y defined on an interval $[0, \frac{\pi}{2}]$ with boundary conditions $y(a) = y(b) = 0$, is

$$y(x) = -x\cos x + (\sin x)(\ln \sin x)$$

[10]

5. The Green's function method can be used to show that if $y_1(x), y_2(x)$ are two solutions of the homogeneous equation, then a particular integral for

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = F(x)$$

is

$$y_p(x) = y_2(x)\int^x \frac{y_1(x')f(x')}{W(x')}dx' - y_1(x)\int^x \frac{y_2(x')f(x')}{W(x')}dx'$$

For the equation below, with the given pair of solutions to the homogeneous equation, find a particular integral

$$\frac{d^2y}{dx^2} - y = \mathrm{sech}x \ ; \ \mathrm{sinh}x, \mathrm{cosh}x$$
$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x\ln x \ ; \ x, x^2$$

[6+6]