PHY218 : Homework 6

1. An oscillating pendulum making small oscillations, in the presence of air resistance, is an example of a *damped harmonic oscillator*. Another example is a mass attached to a spring making small oscillations in the presence of friction. The relevant equation is

$$\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + \omega^2 x = 0$$

If the friction term is sufficiently small, we expect oscillatory motion described by sines and cosines, or equivalently exponentials with imaginary argument. Find these oscillatory solutions, while deriving the condition of smallness of the friction term as a condition on β and ω. [12]

2. An inhomogeneous differential equation takes the form

$$\mathcal{L}y(x) = F(x)$$

where $\mathcal{L}$ is a linear differential operator. Examples of $\mathcal{L}$ are

$$\frac{d^2}{dx^2} + P(x)\frac{d}{dx} + Q(x)$$
$$\frac{d}{dx} + p(x)$$

The corresponding homogeneous equation is

$$\mathcal{L}y(x) = 0$$

If $y_p(x)$ is a solution of the inhomogeneous equation, and $y_c(x)$ is a solution of the homogeneous equation, show that $y = y_p + y_c$ is a solution of the inhomegeneous equation.

Explain why a fixed function $y_p(x)$ allows a most general solution of the inhomogeneous equatio of the form $y = y_p + y_c$.

[6+6]

3. Consider the Hermite equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2\alpha y = 0$$

Take an ansatz $y = \sum_{n=0}^{\infty} a_n x^{k+n}$. Substitute into the equation.

Show that the indicial equation is $k(k-1) = 0$.

Derive the recursion relation :

$$a_{m+2} = 2a_m\frac{(k+m-\alpha)}{(m+k+1)(m+k+2)}$$

Show that the general solution takes the form

$$\begin{aligned} y(x) = \quad & A \left[1 + \frac{2(-\alpha)x^2}{2!} + \frac{2^2(-\alpha)(-\alpha+2)x^4}{4!} + \cdots\right] \\ + \quad & Bx\left[1 + \frac{2(-\alpha+1)x^2}{3!} + \frac{2^2(-\alpha+1)(-\alpha+3)x^4}{5!} + \cdots\right] \end{aligned}$$

What values of α lead to a truncation of the infinite series for the $k = 0$ solutions to finite series (polynomials)? What values lead to such a trunction for the $k = 1$ solutions?

[4+6+6+4]

4. The Legendre equation is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

Substitute the power series ansatz $y = \sum_{l=0}^{\infty} a_l x^{k+l}$.

a) Show that the indicial equation is

$$k(k-1) = 0$$

b) Using $k = 0$, obtain a series of even powers of x

$$\begin{aligned} y_{even} &= a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 + \cdots\right] \\ \text{where } a_{m+2} &= \frac{m(m+1) - n(n+1)}{(m+1)(m+2)}a_m \end{aligned}$$

c) Using $k = 1$, obtain a series of odd powers of x,

$$\begin{aligned} y_{odd} &= a_0\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 + \cdots\right] \\ \text{where } a_{m+2} &= \frac{(m+1)(m+2) - n(n+1)}{(m+2)(m+3)}a_m \end{aligned}$$

d) Which values of n produce a polynomial solution (finite series) from the even solution and the odd solution respectively ?

[4+6+6+4]