PHY218 : Homework 4

1. Show that the inner product

$$(f,g) = \int f^*(x)g(x)w(x)dx$$

obeys

$$\begin{aligned}(f,g) &= (g,f)^* \\ (f,\lambda g) &= \lambda(f,g) \\ (\lambda f,g) &= \lambda^*(f,g)\end{aligned}$$

Make clear in which steps you need to require that the weight function $w(x)$ is real. [2+3+3]

2. Consider functions $\{f_1, f_2, f_3\} = \{1, x, x^2\}$. We define an inner product

$$(f,g) = \int_{-1}^{1} f(x)g(x)dx$$

Calculate the 9 inner products (f_i, f_j).

Now consider another set of functions

$$\begin{aligned}f'_1 &= \frac{f_1}{\sqrt{2}} \\ f'_2 &= \sqrt{\frac{3}{2}} f_2 \\ f'_3 &= (f_3 - \frac{1}{3})\sqrt{\frac{45}{8}}\end{aligned}$$

Show that this new set forms an orthonormal basis for polynomials of degree at most two. [6+4]

3. Prove that the functions $\tilde{e}_k(x) = \sin(\frac{2\pi kx}{L})$ are periodic, i.e

$$\tilde{e}_k(x+L) = \tilde{e}_k(x)$$

for $k = 1, 2, 3...$

Prove that the functions $e_k(x) = \cos(\frac{2\pi kx}{L})$ are periodic for $k = 0, 1, 2, \cdots$.

With the inner product

$$(f,g) = \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)g(x)dx$$

show that

$$
\begin{aligned}
(e_k, e_l) &= \mathcal{N}_k \delta_{k,l} \\
(\tilde{e}_k, \tilde{e}_l) &= \tilde{\mathcal{N}}_k \delta_{k,l} \\
(e_k, \tilde{e}_l) &= 0
\end{aligned}
$$

where $\mathcal{N}_k, \tilde{\mathcal{N}}_k$ are normalization constants you should determine.

Prove that the set $\{e_0, e_1, \tilde{e}_1, e_2, \tilde{e}_2, \cdots\}$ forms a linearly independent set of periodic functions of period L. [3+3+9+5]

4. Consider a periodic function $f(x)$ with period 2π defined as follows. In the region $-\pi \leq x \leq \pi$, define it as

$$f(x) = x^2$$

For all x outside the range $-\pi \leq x \leq \pi$ define it by the periodicity condition $f(x+2\pi) = f(x)$. Sketch a graph of the function.

Determine the coefficients $a_0, a_1, \tilde{a}_1, a_2, \tilde{a}_2, \cdots$ in the expansion

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \tilde{a}_k \sin(kx) + a_k \cos(kx)$$

using the formulae

$$
\begin{aligned}
a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx \\
\tilde{a}_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx
\end{aligned}
$$

Hence, prove that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

[4+6+4]

Hint Have a go based on what we have done in class. If you are getting stuck, consult Riley-Hobson-Bence, Chapter 12. There are enough similar problems there to help you solve this.