PHY218 : Homework 3

1. Given a vector space over $\mathbb{C}$ with an inner product, where $(\mathbf{v}, \mathbf{w})$ denotes the inner product of the vectors $\mathbf{v}$ and $\mathbf{w}$, a linear operator H is said to be hermitian if

$$(\mathbf{v}, H\mathbf{w}) = (H\mathbf{v}, \mathbf{w})$$

An eigenvector $\mathbf{v}$ of H, with eigenvalue λ obeys the equation

$$H\mathbf{v} = \lambda\mathbf{v}$$

Prove that eigenvalues of hermitian operators are real, and that two eigenvectors with distinct eigenvalues are orthogonal. [5+5]

2. The matrix elements of an operator H with respect to a basis $\{\mathbf{e}_i\}$ are defined by

$$H\mathbf{e}_i = \sum_j H_{ji}\mathbf{e}_j$$

Given the definition of a hermitian operator H in question 1, show that its matrix elements in an orthonormal basis obey the equation $H_{ij} = H_{ji}^*$. [6]

3. In an orthonormal basis $\{\mathbf{e}_i\}$, a vector has an expansion $\mathbf{v} = \sum_i v_i\mathbf{e}_i$ and the linear operator H had an expansion $H = \sum_j H_{ji}\mathbf{e}_j$.

Show that

$$H\mathbf{v} = \sum_i \sum_j H_{ij} v_j \mathbf{e}_i$$

Use this to show that the eigenvalue equation $H\mathbf{v} = \lambda\mathbf{v}$ can be expressed in terms of the matrix elements H_{ij} and the components v_i in the form

$$\sum_j H_{ij} v_j = \lambda v_i$$

Remark : If the components v_i are arranged in a column, the left hand side is just expressing matrix multiplication of the matrix H from the left with the column vector. [5+5]

4. An orthonormal basis $\{\mathbf{e}_i\}$ is related to another orthonormal basis $\{\mathbf{e}'_i\}$ by a matrix U as follows.

$$\mathbf{e}'_i = \sum_j U_{ji}\mathbf{e}_j$$

Show that $UU^\dagger = 1$. [5]

5. Diagonalizing an $N \times N$ matrix H involves writing it as $H = UDU^\dagger$ where D is a diagonal matrix, with diagonal elements equal to the eigenvalues of the matrix H, and U is a unitary matrix.

We may write

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & 0 & \lambda_3 & \cdots & 0 \\ & & & \ddots & \\ \cdots & \cdots & \cdots & \cdots & \lambda_N \end{pmatrix}$$

Assuming all the eigenvalues are non-zero, write an expression for the inverse matrix D^{-1} in terms of λ_i.

Prove that $H^{-1} = UD^{-1}U^\dagger$.

[5+5]