PHY218 : Mathematical Techniques 3 - Homework 2

1. As explained in the lectures, sets of basis vectors are not unique. A vector $\mathbf{v}$ can be written as

$$\mathbf{v} = \sum_i v_i \; \mathbf{e}_i = \sum_i v'_i \; \mathbf{e}'_i$$

In an N-dimensional space, any set of N linearly independent vectors forms a basis.

Consider the vector

$$\mathbf{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \in \mathbb{R}^2$$

Find the components v_1, v_2 of the vector $\mathbf{v}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ where

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;,\; \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Now find the components v'_1, v'_2 of the vector $\mathbf{v}$ in the basis $\{\mathbf{e}'_1, \mathbf{e}'_2\}$

$$\mathbf{e}'_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \;,\; \mathbf{e}'_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

[4+6]

2. For two vectors in $\mathbb{C}^N$,

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_N \end{pmatrix} \qquad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \cdots \\ w_N \end{pmatrix}$$

the standard inner product is

$$(\mathbf{v}, \mathbf{w}) = \sum_{i=1}^{N} v_i^* w_i$$

Show that the following properties of inner products hold :

$$(\mathbf{v}, \lambda\mathbf{w}) = \lambda(\mathbf{v}, \mathbf{w}) \text{ for any } \lambda \in \mathbb{C}$$
$$(\lambda\mathbf{v}, \mathbf{w}) = \lambda^*(\mathbf{v}, \mathbf{w})$$
$$(\mathbf{v}, \mathbf{w}) = (\mathbf{w}, \mathbf{v})^*$$

[9]

3. A is a linear operator in a vector space V. There is a basis set $\{\mathbf{e}_i\}$. The vector $A\mathbf{e}_i$ can be expanded in the basis. Denote the coefficients appearing in the expansion by A_{ji} and write the expansion

$$A\mathbf{e}_i = \sum_{j=1}^{N} A_{ji}\mathbf{e}_j$$

The A_{ji} are called the matrix elements of A in the basis $\{\mathbf{e}_i\}$.

Explain the equation

$$(AB)\mathbf{e}_i = \sum_j (AB)_{ji}\mathbf{e}_j$$

[3]

Derive

$$(AB)\mathbf{e}_i = \sum_j \sum_k A_{kj}B_{ji}\mathbf{e}_k$$

[6]

Use these equations to show that the matrix elements of the linear operator AB are obtained from the matrix elements of A, B by the formula for matrix multiplication.

[5]

4. Using the standard inner product in $\mathbb{C}^N$ calculate

- the norms of the three vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} \ , \ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1+2i \end{pmatrix} \ , \ \mathbf{v}_3 = \begin{pmatrix} e^{\frac{i\pi}{3}} \\ 3e^{\frac{i\pi}{4}} \\ 4 \end{pmatrix}$$

[2+2+3]

- the inner product $(\mathbf{v}, \mathbf{w})$ for the pair

$$\mathbf{v} = \begin{pmatrix} 2 \\ -i \\ \sqrt{2} \end{pmatrix} \ , \ \mathbf{w} = \begin{pmatrix} i\sqrt{3} \\ 1 \\ 1 \end{pmatrix}$$

[4]

5. The inverse of an $N \times N$ matrix A is denoted by A^{-1} and obeys $AA^{-1} = A^{-1}A = \mathbf{1}$ where $\mathbf{1}$ is the identity matrix. A^T is the transpose of A. $A^\dagger$ is the hermitian conjugate. Prove the following (no need to use index notation).

$$(AB)^{-1} = B^{-1}A^{-1}$$
$$(A^T)^{-1} = (A^{-1})^T$$
$$(A^\dagger)^{-1} = (A^{-1})^\dagger$$

[4+4+4]