PHY218 : Exercise 6

1. **Proof by induction.**

The solution of the Hermite differential equation by the series method encounters the recursion relation :

$$a_{m+2} = 2a_m \frac{m-\alpha}{(m+1)(m+2)}$$

a) Use the recursion relation to calculate $a_2, a_4, a_6 \cdots$ in terms of a_0.

b) For a general even number $2j \geq 2$, the coefficient is

$$a_{2j} = a_0 \frac{2^j(-\alpha)(-\alpha+2)\cdots(-\alpha+2j-2)}{(2j)!} \tag{1}$$

We will prove (1) by the method of induction. Assuming the equation is true for some fixed $j = J$, show, using the recursion relation that it is also true for $j = J+1$.

Now use the above result you derived, and the fact that it is true in cases $2j = 2$ to explain why the (1) is true for all positive integers j. [6+6+4]

2. For the homogeneous equation with constant coefficients

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$$

show that the ansatz $y = e^{\lambda x}$ leads to the *auxiliary equation*

$$\lambda^2 + P\lambda + Q = 0$$

Hence find the most general solution for the case $P^2 \neq 4Q$, givin an expression for $y(x)$ containing the parameters P, Q.

In the case $P^2 = 4Q$, show that the most general solution has the form

$$y = Ae^{\lambda x} + Bxe^{\lambda x}$$

Write λ in terms of P in this case. [8+4]

3. The Wronskian of two functions $f_1(x), f_2(x)$ is defined as

$$W(f_1, f_2) = f_1 f_2' - f_1' f_2 \tag{2}$$

If the Wronskian is not zero in an interval, then the functions are linearly independent.

Show that the e^{-x} and xe^{-x} are linearly independent over the interval $[0, \infty]$. [6]

Remark This exercise sheet is short. If you have time left, you are encouraged to discuss any questions regarding earlier exercise sheets or homeworks with the instructor.