PHY218 : Exercise 4

1. A useful way to determine if a set of vectors is a linearly independent set is to construct a matrix with rows consisting of the vectors and to calculate the Rank of the matrix. The Rank is defined as the number of linearly independent rows of a matrix, which can be proved to be equal to the number of linearly independent columns. This is sometimes expressed as *column rank equals row rank.*

With two vectors $(1,2),(3,4)$ you can build the matrix $M = \{\{1,2\},\{3,4\}\}$ in Mathematica. You can use the command MatrixForm to display it as a matrix. The command MatrixRank calculates the rank of the matrix. Read the documentation about the relevant commands.

Determine if the following sets of vectors are linearly independent.

$$\begin{aligned}
&\mathbf{v} = (1,2)\ ,\ \mathbf{w} = (3,4)\\
&\mathbf{v} = (1,2)\ ,\ \mathbf{w} = (2,4)\\
&\mathbf{u} = (-1,2,4)\ ,\ \mathbf{v} = (0,1,0)\ ,\ \mathbf{w} = (4,7,1)\\
&\mathbf{u} = (1,3,6)\ ,\ \mathbf{v} = (1,-1,1)\ ,\ \mathbf{w} = (-2,2,-2)\ ,\ \mathbf{x} = (1,0,0)
\end{aligned}$$

[3+3+4+4]

2. Take a pair of 3×3 matrices A, B. You can choose the numbers that go in each matrix. Let some of the numbers be complex. Illustrate the identities

$$\begin{aligned}
(AB)^T &= B^T A^T\\
(AB)^\dagger &= B^\dagger A^\dagger\\
(AB)^* &= A^* B^*
\end{aligned}$$

Useful commands will be Dot, Transpose, Conjugate, ConjugateTranspose. [5+5+5]

3. Again with matrices of your choice illustrate

$$Det(AB) = DetADetB$$

[5]

4. Choose matrices A, B with non-zero determinant and illustrate

$$(AB)^{-1} = B^{-1}A^{-1}$$

[5]

5. Take the Pauli matrix $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Calculate for $\theta = \frac{\pi}{6}, \frac{\pi}{4}$ the exponentials $U(\theta) = e^{i\theta\sigma_x}$. Use the command MatrixExp. Show that $UU^\dagger = 1$. [5+5]

Remark : The Exponential of a matrix A is defined by a series

$$e^A = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

Optional : Show, without mathematica, that

$$e^{i\theta\sigma_x} = \cos\theta + i\sigma_x \sin\theta$$

and that if $A^\dagger = A$ and $U = e^{iA}$ then

$$UU^\dagger = 1$$