MATH 116.3 (02) T2, 2004–05: CALCULUS II

TEST ONE – SAMPLE SOLUTIONS

Each question is worth ten points.

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1. Find the value of the sum. Express the result as a factored polynomial in n:

$$\sum_{i=1}^{n} i(i+1)$$

Solution:

$$\begin{aligned}\sum_{i=1}^{n} i(i+1) &= \sum_{i=1}^{n}(i^2+i) = \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \\ &= n(n+1)\left[\frac{2n+1}{6} + \frac{3}{6}\right] = \frac{1}{6}n(n+1)(2n+1+3) \\ &= \frac{1}{3}n(n+1)(n+2)\ .\end{aligned}$$

2. Consider the function $f(x) = e^x$ on the interval $[0,1]$. Express the area under the graph as a limit. (Do not evaluate the limit.)

Solution: We divide the interval $[0,1]$ into n equal subintervals of width $\Delta x = \frac{1}{n}$. Choosing right endpoints, we obtain the Riemann sum

$$\Delta x \sum_{i=1}^{n} e^{x_i}\ ,$$

where

$$x_i = 0 + i\Delta x = \frac{i}{n}\ .$$

Hence, the area under the graph of $f(x) = e^x$ is

$$\lim_{n\to\infty} \Delta x \sum_{i=1}^{n} e^{i/n}\ .$$

3. Evaluate the Riemann sum for $f(x) = 1 - x^2$ on the interval $[0,2]$ with 4 subintervals using right endpoints as sample points.

Solution: We divide the interval $[0,2]$ into 4 equal subintervals of width $\Delta x = \frac{2}{4} = \frac{1}{2}$. The four right endpoints of the subintervals are $\frac{1}{2}$, 1, $\frac{3}{2}$ and 2. The corresponding Riemann sum is

$$\begin{aligned}\Delta x \sum_{i=1}^{4} f(x_i) &= \frac{1}{2}\left(f(\tfrac{1}{2}) + f(1) + f(\tfrac{3}{2}) + f(2)\right) \\ &= \frac{1}{2}\left(1 - \frac{1}{4} + 1 - 1 + 1 - \frac{9}{4} + 1 - 4\right) = -\frac{7}{4}\ .\end{aligned}$$

4. Express the definite integral as a limit and evaluate the limit:

$$\int_0^2 (1 - x^2)\, dx$$

Solution: With $\Delta x = \frac{2-0}{n} = \frac{2}{n}$ and right endpoints $x_i = 0 + i\,\Delta x = i\,\frac{2}{n}$,

$$\begin{aligned}
\int_0^2 (1 - x^2)\, dx &= \lim_{n\to\infty} \Delta x \sum_{i=1}^{n} (1 - x_i^2) = \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left(1 - \left(i\,\frac{2}{n}\right)^2\right) \\
&= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left(1 - i^2 \frac{4}{n^2}\right) = \lim_{n\to\infty} \frac{2}{n} \left[\sum_{i=1}^{n} 1 - \frac{4}{n^2} \sum_{i=1}^{n} i^2\right] \\
&= \lim_{n\to\infty} \frac{2}{n} \left[n - \frac{4}{n^2} \frac{n(n+1)(2n+1)}{6}\right] = \lim_{n\to\infty} \left[2 - \frac{4(n+1)(2n+1)}{3n^2}\right] \\
&= 2 - \lim_{n\to\infty} \frac{4}{3} \frac{n+1}{n} \frac{2n+1}{n} = 2 - \lim_{n\to\infty} \frac{4}{3} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right) \\
&= 2 - \frac{4}{3} \cdot 1 \cdot 2 = -\frac{2}{3}\,.
\end{aligned}$$

5. Find the derivative of the function:

$$g(x) = \int_0^{x^2} \sqrt[3]{t+1}\, dt$$

Solution: We set $u = x^2$. Then by Part 1 of the Fundamental Theorem of Calculus, together with the chain rule,

$$\begin{aligned}
\frac{d}{dx} \int_0^{x^2} \sqrt[3]{t+1}\, dt &= \frac{d}{dx} \int_0^{u} \sqrt[3]{t+1}\, dt = \frac{d}{du} \left(\int_0^{u} \sqrt[3]{t+1}\, dt \right) \frac{du}{dx} \\
&= \sqrt[3]{u+1} \cdot 2x = 2x\sqrt[3]{x^2+1}\,.
\end{aligned}$$

6. Find the general indefinite integral:

$$\int (7 - 2x^2 + 3x^3)\, dx$$

Solution:.
Use the power rule for each term to get:

$$\int (7 - 2x^2 + 3x^3)\, dx = 7x - \frac{2}{3}x^3 + \frac{3}{4}x^4 + C\,.$$

7. Find the general indefinite integral:

$$\int \frac{x^3}{x^2+1}\, dx$$

Solution: Write the integral as

$$\int \frac{x^3}{x^2+1}\,dx = \int \frac{x^2 \cdot x\,dx}{x^2+1}\,.$$

Let $u = x^2+1$, then $du = 2x\,dx$ and $x^2 = u-1$. So we obtain

$$\begin{aligned}
\int \frac{x^2 \cdot x\,dx}{x^2+1} &= \frac{1}{2}\int \frac{(u-1)\,du}{u} \\
&= \frac{1}{2}\int \left(1-\frac{1}{u}\right)\,du \\
&= \frac{1}{2}(u - \ln|u|) + C \\
&= \frac{1}{2}(x^2+1-\ln(x^2+1)) + C\,.
\end{aligned}$$

In the last step we removed the absolute value bars since $x^2+1>0$ for all x.

8. Evaluate:

$$\int_{-1}^{1} x\sqrt[7]{2+\cos x^4}\,dx$$

Solution: For $f(x) = x\sqrt[7]{2+\cos x^4}$, we have

$$f(-x) = -x\sqrt[7]{2+\cos(-x)^4} = -x\sqrt[7]{2+\cos x^4} = -f(x)\,.$$

Hence, this is an integral of an odd function over a symmetric interval, so

$$\int_{-1}^{1} x\sqrt[7]{2+\cos x^4}\,dx = 0\,.$$

9. Determine the area of the region enclosed by the two curves:

$$y = x^2\,, \quad y = x^4$$

Solution: First, determine the intersection points by solving the equation $x^2 = x^4$.

$$\begin{aligned}
x^2 - x^4 &= 0 \\
x^2(1-x^2) &= 0 \\
x^2(1-x)(1+x) &= 0\,.
\end{aligned}$$

So the intersection points are $x = 0, -1, 1$. The formula for calculating the area between these two curves is

$$A = \int_{-1}^{1} |x^4 - x^2|\,dx$$

To remove the absolute value bars, use the fact that $x^4 < x^2$ when $-1 < x < 1$. So,

$$\begin{aligned}\int_{-1}^{1} |x^4 - x^2|\, dx &= \int_{-1}^{1} (x^2 - x^4)\, dx \\ &= 2\int_{0}^{1} (x^2 - x^4)\, dx \\ &= 2\left(\frac{1}{3}x^3 - \frac{1}{5}x^5\right)\Bigg|_0^1 \\ &= 2\left(\frac{1}{3} - \frac{1}{5}\right) \\ &= \frac{4}{15}\,.\end{aligned}$$

In the second equality, we have used the fact that $f(x) = x^2 - x^4$ is an even function.

10. Determine the area of the region enclosed by the two curves:

$$y^2 = x + 3\,, \quad x = 1$$

Solution: Note that the first curve is not given as a function of x, but of y, so that the integration will have to be done in y. Find the intersection points of $x = y^2 - 3$ and $x = 1$ by solving

$$\begin{aligned} y^2 - 3 &= 1 \\ y^2 &= 4 \\ y &= \pm 2\,. \end{aligned}$$

The graph of $x = y^2 - 3$ is a parabola with vertex at $(-3, 0)$ opening to the right, so the area between the curves is

$$\begin{aligned} A &= \int_{-2}^{2} |1 - (y^2 - 3)|\, dy \\ &= \int_{-2}^{2} |1 - y^2 + 3|\, dy \\ &= 2\int_{0}^{2} (4 - y^2)\, dy \\ &= 2\left(4y - \frac{1}{3}y^3\right)\Bigg|_0^2 \\ &= 2\left(8 - \frac{8}{3}\right) \\ &= \frac{32}{3}\,. \end{aligned}$$

In the third step we removed the absolute value bars since $4 - y^2 \geq 0$ for $0 \leq y \leq 2$, and we also used the fact that $f(y) = 4 - y^2$ is an even function.

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